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ABSTRACT

This report presents 23 papers prepared for the second Anglo-Soviet Seminar on Mathematics Education, held in Moscow in September 1982. Notes on the points highlighted in the subsequent discussion are also included. Papers are grouped into 10 sections on aspects of mathematics education: problems of methodics in introducing universal secondary education, major current problems, aims, achieving a balance, developmental work, how to achieve what is planned, the new geometry course in the USSR, some specific issues, methods of assessment, and training mathematics teachers in the USSR. In some cases, at least one British and one Soviet paper were presented on the same topic. (MNS)

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Mathematics education

the report of the second Anglo-Soviet Seminar
held in Moscow at the Institute for Scientific
Research into Content and Methods of Instruction,
Academy of Pedagogical Sciences of the USSR

September 20-30, 1982

under the auspices of the Ministry of Education
of the USSR and the British Council

Chairman Professor V M Monakhov
Leader of British delegation Dr T J Fletcher,
CBE, HMI

Editor Bryan Wilson

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PREFACE

The first Anglo-Soviet Seminar on mathematics education took place at St Antony's College, Oxford, in September 1981, under the terms of the Anglo-Soviet Agreement on Relations in the Science, Educational and Cultural fields. Ten leading British mathematics educators met with six Soviet participants for an intensive programme on information sharing and discussion covering the teaching and learning of mathematics at all school levels in the two countries. At the close of this seminar, much satisfaction was expressed by all the participants as to its professional value. Much had been learned, and various issues of importance to both countries had been identified. A report of this first Seminar is available from Consultancies Group, Science, Technology and Education Division, the British Council, 10 Spring Gardens, London SW1A 2BN.

Following the first Seminar, the Soviet Ministry of Education issued an invitation to a second Seminar, to take place in Moscow. This invitation was warmly welcomed by the British Council, and the same British team visited Moscow from 20 to 30 September 1982 to take part in the second Seminar, on which this publication reports.

The Seminar took place in the Institute for Scientific Research into Content and Methods of Instruction, of the Academy of Pedagogical Sciences of the USSR, under the Chairmanship of Professor V M McNakhov, Director of the Institute. The fact that both he and Mr V V Firsov, Head of the Institute's Mathematics Teaching Laboratory, also took part in the first Seminar was a great help in ensuring that the work built on the foundations laid at Oxford. A list of participants will be found at the end of this report.

As at Oxford, the programme consisted of pre-circulated papers which were introduced by their authors, followed by time for questions and discussion. These papers are reproduced in this report, together with notes on the points highlighted in the subsequent discussion.

One morning was given over to school visits. Two of the British party visited experimental primary classes and the special mathematics and science school associated with Moscow State University, while the rest saw various classes in progress at School No 28. Notes on these school visits are included towards the end of this report. We would like to thank the Principals, teachers and children who welcomed us so graciously to their schools.

A comprehensive exhibition of both Soviet and British mathematics education texts and other books was mounted to coincide with the Seminar. We also appreciated an evening meeting with the members of the Editorial Board of "Qvant", the splendid mathematics and science monthly journal designed for school pupils.

Our Soviet hosts arranged a most enjoyable social programme, including visits to the Moscow State Ballet, to the circus and to the Tretykovskaya Art Gallery, tours of the city and of the Kremlin, and an outing to the Archangelskoe Country House Museum. The latter was followed by a gargantuan picnic in a wood at which we learned some Russian games and our Soviet friends some English songs. We would like to record our appreciation of the generous hospitality extended to us on these and a number of other occasions.

We wish to express our particular thanks to the Ministry of Education of the USSR for the invitation to the Seminar, to Professor Monakhov for his Chairmanship, and to his colleague Mr Firsov, who bore the main burden of the organisation of the Seminar. We also wish to thank the other members of the Soviet team for their welcome and their helpfulness, and not least to the interpreters and the guides without whom we would have been lost both linguistically and geographically. We left Moscow knowing that we were leaving not just professional colleagues but also friends. It is our sincere hope that the relationships established through these two Seminars will enable continuing collaborative work between our two countries in the field of mathematics education. We are confident that this will indeed prove to be the case.

Editor's note on the Soviet use of certain key words.

Algorithm

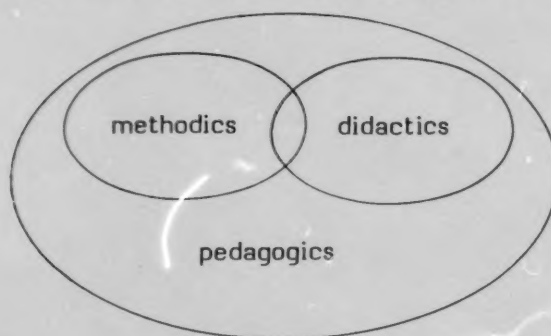
This appears to have a wider connotation than is usual in Britain. Applied to problem-solving, it refers to a systematic way of teaching children to tackle certain kinds of problem, and includes aspects of what in Britain would be described as a problem-solving strategy, as well as incorporating the standard mathematical processes involved. It can also be applied to, eg, learning a piece of mathematics, or acquiring a concept. In each case, careful analysis is necessary to discover the steps involved in the process, these steps then forming the algorithm.

Soviet practice is to study textbooks and other materials to ensure that they conform to such algorithms.

Methodics

The principles of teaching a particular subject. It is much broader than the English use of 'methods'.

The following diagram was produced by the Soviet participants in an attempt to clarify the understanding of the British participants:



Pedagogics = child study in general

Didactics = general principles of teaching

Methodics = principles of teaching a particular subject. It is not a sub-set of didactics, as it also takes account of the nature of the particular subject.

Propaedeutics

The informal introduction of a mathematical concept or topic in preparation for its more systematic development later in the school course. It is used to a very limited extent in the Soviet mathematics curriculum. It is, of course, the essence of the 'spiral approach' widely adopted in Britain.

OPENING SESSION

The Seminar took place in the Presidium of the Academy of Pedagogical Sciences of the USSR. It was formally opened by the Vice-President of the Academy, Academician I D Zverev. In welcoming the British delegation, Academician Zverev said that the Soviet delegation to Oxford had brought back to Moscow excellent reports of the first Anglo-Soviet Seminar on Mathematics Education. Mathematics is a good theme for cooperation between countries which have different social and political systems. He affirmed his belief that further collaborative work would be of mutual benefit to the two countries, and in particular to the work of the Mathematics Section of the Academy. Since the Academy has the responsibility of developing and deciding on school curricula and teaching materials throughout the country's entire school system, the education of every Soviet child could be affected by the work of the Seminar. He expressed his gratitude to the Department of Education and Science and to the British Council for their assistance in organising this meeting.

Academician Zverev then handed over to the Seminar Chairman, Professor V M Monakhov, to present the first Seminar paper.

Session 1.

PROBLEMS OF METHODICS IN INTRODUCING UNIVERSAL SECONDARY EDUCATION.

Professor V M Monakhov, Director of the Institute for Scientific Research into Content and Methods of Instruction, Academy of Pedagogical Sciences of the USSR.

1. The 60th anniversary of the founding of the Soviet state is being marked in our schools by the introduction of universal compulsory secondary education. The Soviet school has arrived at 10-year universal compulsory secondary education by passing through several stages: universal primary, seven year, eight year and finally ten year education. The people of our multi-national country rightly regard the introduction of universal secondary education as an important step towards the fulfilment of one of man's most important rights - the right to education. Article 45 of the Constitution of the USSR states: "This right is fulfilled through the fact that all types of education are free, through the existence of universal secondary education for young people, and through the widespread development of professional-technical, secondary-special and higher education".

In the Soviet Union no constitutional right of man is simply a declaration. Its fulfilment is guaranteed by the state and brought about by a wide range of social, cultural and economic measures. The universal nature of education means that education becomes an obligatory element in the 'training-for-life' of all young citizens of the country without exception, regardless of sex, nationality, race, property and other distinctions.

2. The transition to universal secondary education meant that pedagogical science faced the necessity of solving a series of new problems of methodics. In the first instance it was necessary to find a solution to the problem of a single level of compulsory training for all pupils receiving secondary education. As is well known, there are in our country three types of secondary educational institutions: general schools, secondary vocational schools and secondary special schools each of which is required to provide the necessary level of general educational training.

The key element as regards the practical realisation of universal secondary education is the identification of a single core of general educational training which is compulsory for all those who pass through the system of secondary education, and the 'fixing' of that core in the form of normative and methodic documentation which indicates the minimal level of training necessary for work and further studies: a level which the state guarantees to every young person.

3. The basis for the solution of the problem of identifying a scientifically well-founded single level of content for universal secondary education is the following conception:

(a) The system of universal education now existing in schools, secondary-special and vocational schools achieves the aims of secondary education as formulated in the programme of the Communist Party of the Soviet Union. "Secondary education must provide a thorough knowledge of the bases of the sciences, mastery of the principles of the Communist world-view, training for work and polytechnical training in accordance with the increasing development of science and technology, taking into account the needs of society and the abilities and wishes of pupils: it must also provide moral, aesthetic and physical training for the growing generation" (Programme of the CPSU. - Moscow, Politizdat, 1974, p. 123).

(b) In determining a single level it is necessary to preserve the character of the general schools, secondary vocational and secondary special schools: the general schools are concerned with the pre-professional training of pupils, while the vocational schools and special schools train qualified workers, for work at the middle level of various branches of the national economy, ie they are concerned with professional (special) training.

(c) For a standard of general education comparable to the standards provided by the secondary vocational schools and secondary special schools the obvious choice is the general school. The reason for this lies in the fact that this is the most widespread system; it takes in 50-60% of eight-year-school pupils and has justified itself through long and all-round practical experience.

(d) In order to establish a single level of content for general education the following conditions must be taken into account:

- the nomenclature of general educational subjects in all three types of educational institutions must coincide;
- the basic content of humanities subjects must be one and the same;
- the system of basic concepts, laws, theories and methods of science which constitutes the content of scientific and mathematical disciplines must be identical in all three types of school;
- the amount of teaching time devoted to the study of general-educational subjects must be sufficient to ensure mastery of the content selected for study and must not differ markedly among the various types of secondary educational institutions.

(e) The realisation of the idea of a single level of teaching for scientific and mathematical disciplines does not presuppose the mechanical 'levelling-out' of general educational training; it is organically linked to a differentiated approach to the selection of content, since it must reflect the specific nature of each type of secondary educational institution.

4. One complex problem of methodics is the 'fixing' of a single level in the form of normative and methodic documentation.

The general-educational subjects in all three types of secondary educational institutions rest on a single base - fundamental concepts, laws, theories, important facts and methods of inquiry. The academic material, which forms a single base for general-educational training and which is included in the content of general disciplines in all three types of secondary educational institution, constitutes an invariable component in the content of general education, in which the single level is 'fixed'.

Thus for example the invariable component of mathematics education is defined by the necessity of forming an idea of the basic laws and methods of acquiring knowledge applied by mathematical science, by the logic and structure of the science, and by its language, and by the necessity of acquainting pupils with the role of mathematics in constructing a complete scientific picture of the world. A study of mathematics facilitates the development of the knowledge and skills necessary for practical activity and further studies.

In defining the invariable component of mathematics education one must take into account the leading role played by mathematics in the development of pupils' powers of logical thought and abstract conceptualisation, of the desire to prove and justify their statements, and of logical intuition and skill in constructing a deductive schema for drawing conclusions. The actual objects of mathematical conclusions and the mathematical rules for constructing them in the clearest and most concise form reveal the mechanisms of logical structure and show how these can be applied.

5. Apart from the invariable part in the content of general-educational subjects there is material the need for which arises out of the concrete methodics adopted in this or that educational institution, and also material essentially linked with technical or special subjects, with training for work and industry, applied and professionally oriented material. All of this constitutes the variable component in the content of an academic subject. The fact that this or that material is assigned to the variable component by no means implies that it is automatically excluded from the syllabuses of general-educational subjects in all educational institutions - a number of circumstances may make it necessary to study this

material in a particular institution, for example, with a view to work-orientation or to establish inter-subject links with professional (specialised) disciplines. Thus, it is possible to establish a single level of content for general secondary education in secondary special and vocational schools, given the division of the content of education into variable and invariable components and given the fact that this is reflected in the documentation of academic syllabuses.

6. The single core of general educational training which has been identified is 'fixed' in the form of so-called core curricula. On the basis of the core curricula which 'fix' the single level, functional syllabuses which take into account the specific nature of individual educational institutions are worked out and introduced into those institutions.

The introduction of the functional syllabuses gave rise to new problems of methodics. The foremost of these were: correction of most of the existing textbooks and handbooks on methodics; the preparation and experimental testing of new textbooks for a number of subjects; the elaboration of methodics guidelines at various levels - from letters - a kind of teachers' first aid for dealing with the immediate introduction of changes into schools - to fundamental methodics for all basic subjects.

An important feature of this work is the emphasis on the identification of what is most important, most essential as regards the methodics aspect of academic training. This presupposes a corresponding differentiation in academic requirements as regards mastery of the primary and secondary academic material.

Experience in the use of the functional syllabuses shows that their positive potential as a means for directing the learning process increases as the requirements that the level of learning must meet are made more concrete. It is well known that teachers show a greater interest in the more concrete types of methodics documents: thematic planning, didactic material, examples of tests and so on. This is no accident: the teacher 'sees' in concrete academic requirements the final outcome of learning, a sort of planned result, and bearing this in mind he or she organises his or her classwork. Therefore the concretisation of knowledge requirements and required skills is something which we regard as providing an important opening for the improvement of syllabuses.

7. The introduction of universal secondary education for pupils in Soviet schools on the basis of single curricula of general educational training by no means implies a forced levelling out in the training of school pupils. Such an understanding of the conception of a single level would totally contradict the fundamental aim of the USSR's educational system - namely the creation of a multi-faceted and harmonious individual. The latter presupposes first and foremost the maximal development of the creative talents and interests of the pupils. The single level is as it were a starting point, a basis for the

organisation of the learning process, in which content, form and teaching methods are oriented towards developing pupils as fully and as variously as possible. This is brought about by various means which include differentiated forms and methods, the differentiation of requirements which pupils must meet, taking into account their interest in the subject. The very structure of the Soviet education system provides for this both in the lessons and outside them by means of the widespread use of various forms of extra-curricular work - subject circles, pupils' associations, olympiads and so on. The optional studies which are run from the 7th class onwards and which pupils follow according to choice are an important means of differentiating the learning process. The main characteristic of this form of learning consists in the fact that every pupil in the school who shows an interest in this or that subject can take part in these courses. The democratic principle of the choice of an option results in the widespread adoption of this form of work. Thus for example up to 2 million pupils in Soviet secondary schools who are interested in mathematics receive additional mathematical training by way of optional courses.

Of course, in the teaching of every subject the transition to universal secondary education gave rise to specific problems. However both for mathematics and for other subjects the central trend in finding a solution to these problems is linked to the identification of the fundamental and most essential part of general-educational training and also to the varied development of pupils' interests, inclinations and creative abilities by means of the use of varied forms of academic work and methods of teaching.

The discussion following Professor Monakhov's presentation centred on the training of mathematics teachers, a theme to which the Seminar returned later in its programme.

In response to a question as to whether teachers' manuals were found to be an effective medium for training new mathematics teachers in the Soviet Union, Professor Monakhov asserted that it was essential for the young teacher to know the textbooks used in schools thoroughly. Furthermore, he must also master the auxiliary aids and manuals available to him, since these are also obligatory documents on the basis of which he must work. The reforms of the 1970s had produced a whole range of manuals on teaching methods, and if left to himself the young teacher would be swamped by them. Consequently the Institute for Scientific Research into Content and Methods of Instruction had analysed all the relevant recent literature, in order to define a limited number of manuals which represent the current philosophy and practice of mathematics teaching. Nevertheless, it was asserted that the priority in initial training of a mathematics teacher must lie with mathematics, rather than with teaching methodology.

All serving teachers are obliged to attend an in-service course at least once every 5 years. Before the introduction of a new textbook, all teachers at the Grade affected attend a 2-4 week course during the previous summer holiday. These courses are organised on a pyramid type system, at National, Republic and Regional levels, with the students at each level becoming the tutors on the courses at the next level. Although there is some danger of distortion of content on the way in such a system, it is the only way to reach every teacher in a vast country; some self-education material reaches teachers directly, for example through the journal "Mathematics at School", and this acts as a check. Further, the Inspectorate of the Ministry of Education ensures that every school holds pre-term conferences, subject by subject, for its teachers every August; all teachers without exception are included. During term time, there are monthly meetings of all the mathematics teachers within each school in "Methods Committees"; at these meetings, articles from the professional journals, results of test papers, and so on are discussed. Each teacher also spends at least 3 lessons per month visiting classes of other teachers. All this helps to ensure uniformity of standards and of methodologies of teaching. No use is made of film as a medium for teacher education, and not much enthusiasm was apparent for the use of visual aids in mathematics teaching or teacher-training.

PROBLEMS OF MATHEMATICS EDUCATION IN ENGLAND

Dr T J Fletcher, CBE, HMI, formerly Staff Inspector (Mathematics)

There is no difficulty in listing problems of mathematical education and so I will limit myself to problems which may have an international interest. In the United Kingdom the school systems of England, Wales, Scotland and Northern Ireland are separately administered, and I am talking about England and not about the rest of the United Kingdom.

Early in 1982 a very important report was published. "Mathematics Counts" was written by a committee of inquiry into the teaching of mathematics in schools set up by central government. This report supplies a detailed analysis of the problems of mathematical education in the country. It is the most important study of its kind since 1912 when the Board of Education published two volumes for the initial meeting of the International Commission for Mathematical Instruction. The present report is known as the Cockcroft Report after the name of the committee's chairman.

I believe that there are several reasons why the report should interest people in other countries. The content of mathematics is much the same internationally and so the problems of one country are often repeated in others. The social history of the Cockcroft Report is particularly interesting and it shows one way of handling public criticism of the educational system in a democratic society. The terms of reference of the Cockcroft Committee were:

To consider the teaching of mathematics in primary and secondary schools in England and Wales, with particular regard to the mathematics required in further and higher education, employment and adult life generally, and to make recommendations.

Therefore the report is particularly concerned with mathematics in relation to the needs of society, and this, in different ways, is the concern of the educational system in all countries.

I will structure this present exposition in a way similar to the report because I think if you can overcome the language difficulties you will find much of interest in the report, and I would like to encourage you to look at it.

The problems of higher education, employment and adult life generally

One problem of schools is to provide the pupils with the qualifications needed to proceed to study in universities and colleges. Since 1945 our system of higher education has expanded greatly, but we are now entering a period of contraction partly because of population changes and partly because of the problems of world economic recession (our share of the world, that is!). There are problems concerned with what mathematics to include in courses for pupils of the age 16-18 and also problems of how best to teach students so that they develop good attitudes to the subject and the ability to study on their own.

The problems of employment and adult life generally are rather different. In preparation for the Cockcroft Report research was undertaken on the mathematics which is used by young people who leave school (16-18) and enter employment. Many industries, such as engineering, use very advanced mathematics, but this is not demanded of the 16-18 year old leaver, and we are not concerned with these higher levels of mathematics here.

During the 1970s there had been much criticism of the inability of school leavers to cope with the mathematics required in employment. The committee found that these criticisms had been considerably exaggerated. In fact at the present time it is very hard to find firmly established complaints from employers except in the engineering industry and to some extent in the retail trades (shop assistants), although even here this problem is found only in the inner cities, especially in London. However, there was a big change in the economic situation in 1975-6. Before that time the number of available jobs exceeded the number of school-leavers, but now the reverse is true.

Studies were undertaken at Bath and Nottingham Universities into the mathematics required in a great variety of jobs. It turns out that the actual skills required are covered by courses in schools. You must remember that there are no national syllabuses in England and that schools have great freedom to decide what to put into their courses. It was not therefore self-evident that school courses were meeting the needs of employers. However, there are many aspects of mathematics as it is actually used in work which schools do not usually consider. Estimation and approximation are extremely important, and in industry mathematics is nearly always concerned with measurements and not simply with numbers. Should school mathematics give more attention to these aspects of the subject?

There is abundant evidence from Inspectors' reports and elsewhere that schools at the moment devote much time to practising numerical skills, but the results are often disappointing. How can schools adapt their teaching so that pupils acquire greater confidence in what they know and greater ability to apply it to everyday matters? There

needs to be a greater awareness of the practical applications of the mathematics taught, more discussion, more simple straightforward mental calculation and more practical work involving measurement. In England employers usually have selection tests for those who work for them. At the moment some employers' tests are very unsatisfactory. The questions are unrealistic and badly related to the requirements of the job.

A lot has been done in recent years to develop better liaison and understanding between schools and industry but there is a constant need to work at the problem. It is difficult for both sides to find time and it is never easy to find the best ways of achieving liaison.

The mathematics used by adults in everyday life

The Cockcroft Committee also commissioned studies of the mathematics used by adults in everyday life. Much was learned about the confusions, the irrational dislike and even the fear which many adults have for mathematics. Many people were willing to help the investigator study some of the problems of education - but when they learned that they would have to answer some simple questions about mathematics they refused to go any further. People were questioned in the street by public opinion poll methods and they revealed some very elementary misunderstandings. It is a problem to teach mathematics in a way which gives people much greater confidence than many achieve at the moment, and to help them make effective use of whatever mathematical skill and understanding they have.

Mathematics in primary schools

I believe that the way to approach the teaching of mathematics in primary schools has been shown repeatedly throughout this century. There has been no shortage of good advice. The problem is to know why teachers do not take the good advice which is available.

The mathematical knowledge and the mathematical self-confidence of many primary teachers is disappointingly low. It is not easy to teach mathematics well. Exposition and the practice of skills is not enough. Children need practical experience, their individual difficulties have to be diagnosed and overcome, and the teacher has to be prepared to discuss ideas as they arise and to follow the different directions which the ideas can take. There are always people (parents, external critics of the educational system, and even some teachers themselves) who are anxious for quick results and to believe that quicker methods can be found. As a result there is always pressure for the style of teaching that is restricted to exposition by the teacher and repetitive practice by the pupil.

The report states that mathematics teaching at all levels should include opportunities for exposition by the teacher, discussion, practical work, consolidation and practice of skills, problem-solving including problems of everyday situations, investigational work. It is a substantial problem to convince teachers and the public that an approach of this breadth is necessary if pupils are going to be able to apply what they learn. Indeed, if I had to identify one problem as more important than any other then I would identify this.

Differences in attainment

One problem which the Cockcroft Committee identified, and to which they attached great importance, is the problem of the "seven year difference". I am very aware that when we discuss this question we are going into an area where English and Soviet educational principles and philosophies are greatly at variance. It is not my intention to say that one is right and the other is wrong, and I wish most of all to talk about problems which we have in common; but if we are to understand one another then we must understand the differences as well as the similarities.

The report says:

"We drew attention... to the 'seven year difference' which (educational tests show) exists among 11 year olds. If we relate this to work in secondary years, it means that the mathematical understanding of some pupils who transfer to secondary school at 11 is likely already to be greater than that of some pupils who have just left school at 16. On the other hand, some of those who arrive at the same time may not, while at school, attain the understanding which some of their fellow 11 year olds already possess. When considering work in the secondary years it is also necessary to remember that pupils learn at very different speeds and that... mathematics is a hierarchial subject."

The report goes on to discuss how these differences in learning speed can be accommodated in the curriculum. This problem arises in the primary school and becomes progressively more acute throughout the secondary school. Note that we are talking about differences in attainment at any given age. One may talk loosely about children having differing abilities - but we do not need to raise the issue of whether children's endowment at birth or their ultimate potentials are the same or different. We are recognising only that at any given age their recent attainments differ, and we are saying that differing attainment implies different needs.

We would see the attempt to vary the curriculum so as to accommodate different levels of attainment as being an application of the principle

"To each according to his needs, from each according to his ability."

Over the last 15 years or so the bipartite (sometimes tripartite) system of grammar schools and modern schools has been almost completely replaced by the system of comprehensive schools. At the same time there has been a great extension of the examination system at the age of 16, so that it now covers the majority of the school population, whereas previously it catered almost exclusively for the grammar schools. This has been accompanied by a very substantial widening of the mathematics syllabus for most pupils; this has meant a substantial attempt to teach the kind of mathematics which was previously taught only to the few to most of the population. (This means perhaps 60% to 90% of the pupils, depending on the particular school.) The Cockcroft Committee believed that for the majority of pupils this broadening of the syllabus (in the sense of a list of topics to be covered) had gone too far and they recommended that the amount of material to be covered should be reduced but that it should be taught by a much greater variety of teaching approaches. This can be described as a reduction of the syllabus but a broadening of the curriculum.

At the moment I believe that England and the USSR are moving in opposite directions in respect of breadth of syllabus, and in the attempt to teach "mathematicians' mathematics" to as many pupils as possible. I believe that the trend advocated in the Cockcroft Report is a reaction to the excessive optimism of the previous two decades. I do not see this particular difference between England and the USSR as indicating a difference in aims - may I suggest as the explanation that we are in opposite phases of the same dialectical process.

To summarise, the proposal is that for many secondary pupils there should be a shorter list of mathematical topics to be taught, but a broader curriculum - that is to say there should be a greater variety of teaching approach, a greater variety of working style in the classroom and greater emphasis on the practical application of the ideas taught.

Before leaving this topic I wish to draw attention to a further difference between our two countries. In England there is a long tradition that examinations at 16 grade pupils; there is essentially a competition for jobs, for places in higher education and for social advantage. Teachers for the most part do not welcome this - neither I suspect do members of the English party visiting you - but we have all of us been successful in this system and

we owe our positions to it. This means that there is in England a tendency for education to magnify children's differences, whereas I believe that in countries of the Warsaw Pact there is a tendency to adopt classroom methods which do not reveal these differences so strongly.

A major problem which we have in mathematical education is certainly to devise suitable courses for the lower achieving pupils of the ages of 13-16. The Cockcroft Report has much to say about this but I do not wish to go into too much detail here.

We also have problems with pupils aged 16-18. An increasing number of pupils in this age group study mathematics. Because the course which leads to university is built around three or perhaps four subjects chosen by the student, when mathematics is chosen it is studied to a comparatively advanced level, and the course involves a wider range of applications to mechanics and/or statistics than is usually the case in other countries. We have problems with these courses but they are comparatively minor. The bigger problem - largely untackled - is designing courses for pupils who do not want such a fully detailed and demanding course in mathematics and who may not go on to university.

Miscellaneous problems

I will mention a few more problems which we might wish to discuss, but I will not explain them at any length.

1. Problems of teacher training

i. What should go into teacher training courses? Remember that our courses are not long compared with those of some other countries.

ii. How can we recruit sufficiently well qualified mathematics teachers when a knowledge of mathematics is a key to so many other professions which (people believe) are better paid.

iii. How can people in education persuade government that the improvement of standards of teaching which we all wish to see will be achievable only if greater investment can be made in in-service training?

2. Problems of the place of women in mathematics (and in society). What brings about the present disparities between the achievement of men and women in the mathematics profession and in the teaching profession? Are the causes of these disparities removable, and should they be removed?

3. What part should microcomputers play in mathematical education over the next few years?
4. How can we inform parents better about the state and intentions of the school curriculum, and how can we involve them in the future reshaping of the curriculum?

In the amplification and discussion of Dr Fletcher's paper, the major theme to emerge was the issue of differing attainments in mathematics between children of the same age and at the same school level. How do you measure these varying attainments, and how do you provide for them?

The proverb "you can lead a horse to water, but you cannot make it drink" was quoted. It was suggested that Soviet work focuses on the question: How much water should we provide for the horses? In Britain the concern is more with: How much water have the horses shown themselves capable of drinking?

In amplification of the Cockcroft "7-year gap" Dr Fletcher explained that one of the very few statutory requirements on the school curriculum in England is that a child must be educated "according to his age, ability and aptitude". Although it may be thought that the examination system determines the curriculum, no examination is compulsory for any individual child, and thus the system does not inhibit the availability of different curricula for different children according to their perceived needs at different times. However, public examinations are norm- rather than criterion-referenced, and criterion-referenced assessment procedures form an area in which Britain should be able to learn from Soviet practice.

Editor's Note. Throughout the seminar it was found that it was more useful to speak about different attainments between children rather than different abilities. The latter would imply certain views of society and of children that the Soviet participants would not necessarily be prepared to accept, whereas different attainments are evident for all to see in either country.

For example, not all Soviet pupils are selected for mathematical Olympiad teams, while the Assessment of Performance Unit (APU) has demonstrated wide variation in attainment within the same age group among English pupils. At the same time, the British participants came to believe that the variation in mathematical achievement found in a typical class in a Soviet school is probably considerably less than would be found in a corresponding "mixed-ability" class in an English school.

THE BASIC TRENDS IN THE IMPROVEMENT OF MATHEMATICS TEACHING IN THE PRESENT DAY SOVIET SCHOOL

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The major reform in school mathematics education which has been carried out in the USSR in recent years set out to remove a certain disproportion between the content of mathematics as an academic discipline taught in general schools and the content of mathematics as a science, and to bring the content of what was taught into line with the needs of modern life and practical requirements. As this reform was carried out, the school mathematics course underwent a fundamental reorganisation (as against the partial reforms of earlier years). The changes affected both teaching content (the exclusion of archaic material, the introduction of new sections) and course structure (changes in the range of mathematical subjects and their functions in relation to the mathematics course as a whole). At the same time the traditional system of teaching, which had previously been oriented to a far greater extent towards the achievement of the specific ends of formation and reinforcement of elementary skills, was also changed.

The new mathematics course for Soviet schools was designed and built as a single organism, beginning with the first class. This was reflected in the important development of links within the subject, which provided a basis for continuity in the teaching of it. The content of the course develops around a small number of content and methodological "threads" running from class to class, which unite the various levels in the formation of interconnected concepts. A course structure of this type facilitates the precise identification of the subject's concept-core, the elements of which form a basis for the study of the remainder of the course content.

The nomenclature of the sections of the new syllabus takes into account the drawbacks of the old course and excludes them. Thus in the new syllabus the old subject "arithmetic" has disappeared, and the study of arithmetic material is carried out in connection with the study of algebraic material (in the primary classes on the level of algebraic propaedeutics, in the middle and senior classes as part of an organic whole). Much more attention is devoted to the study of functions, culminating in the study of basic differential and integral calculus. In the geometry course modern methods of solving geometric problems with the help of geometrical transformations, co-ordinates and vectors are developed. The surface areas and volumes of geometric solids are calculated with the help of integrals.

The course was constructed as a general one - that is to say, the study and mastery of it are assumed to be possible for all pupils in Soviet secondary schools. This required corresponding methodological provision for the education process and for the training of teachers.

As the reforms were being carried out, series of teaching resources were prepared to make it possible to organise the pupils' academic work both in class and at home. For the first time pupils were offered mathematics text books which fulfilled the function of directing the academic process at each of three stages: the study of new material, consolidation, and revision of what had gone before. For the first time a system of exercises is included in these textbooks which is organically linked with the presentation of theory, and which takes account of the propaedeutic, instructional, consolidatory and testing functions. For the first time text books were supplemented by sets of methodological handbooks for teachers, which made possible uniformity of methodological requirements but variety in the forms and methods of teaching, and the differentiation and individualisation of education.

Thus as regards the majority of concepts in the school mathematics course the formal application of the set theory approach leads to the formation of concepts outside the sphere in which they will subsequently be used. If on the other hand the fundamental concepts are formed in the same sphere in which they are to be used then they can be formed at the level of the content and these included naturally in corresponding theories.

An analysis of the content of mathematics text books has shown that an approach on the basis of content could be adopted without making it necessary to alter the textbooks radically and without affecting the basic content. At the same time a rejection of the formal application of the set theory approach makes it possible to simplify the presentation considerably and to lessen the course workload. The adoption of the content-based approach makes it possible to strengthen the applied content of the mathematics course, and the time which is saved by not studying formal constructions can be used instead to form and develop pupils' practical skills. Such a reorganisation is external and does not affect the nomenclature, range and depth of study of the fundamental content of the course. At the present time the reorganisation is being carried out through systematic work on the correction of teaching and methodological handbooks.

The movement in the direction of reforms in mathematical education throughout the world was inspired, as R Thom correctly remarked, not only by ideas about the modernisation of teaching content. Of no less importance were the pedagogical aspirations of the reformers, which were directly connected with a desire to free schools from absurd traditional methods of formal and dogmatic teaching. Throughout the world disciples of

the new pedagogy, inspired by their idea, stepped outside the limitations dictated by the social and cultural realities of the day, over-estimated the possibilities of putting into practice these innovatory pedagogical ideas, and in a number of cases seriously discredited them. We are now returning to an age of sober and reasonable compromises and a search for balanced approaches in the use of new methods, to a re-examination of and a greater respect for the traditional methodological system of mathematics teaching.

The well-defined and consistent structure of the Soviet school is undoubtedly one reason why the extremes of the process described above passed it by. In particular, the most important methodological principle of Soviet pedagogy, which denies the existence of a single ideal teaching method and gives priority to a reasonable combination of methods and devices, was not violated. The conception of the optimisation of teaching methods, to be found in the work of Yu K Banansky, concerns the procedures for choosing methods and devices from the point of view of their adequacy in relation to the overt aims of teaching.

The clear identification of the aims of mathematics teaching makes possible the planning of teaching outcomes, which presupposes the identification in advance of concrete academic results which must be achieved by pupils at any given stage of their education. These results may be differentiated among groups of pupils (poor, average, good) and among different phases of education (lesson, topic, year). The planned outcome of mathematics teaching must satisfy the following conditions:

- a) Specificity This means that teaching outcomes must be formulated in a language which includes or implies corresponding procedures for checking the extent to which they have been achieved. This is a necessary precondition for the possibility of directing the learning process towards planned outcomes.
- b) Realism This means that the performance of tasks must be absolutely within the capabilities of any particular pupil. This condition is particularly important in relation to the content of homework.
- c) Openness This means that the pupil must understand what is expected of him and how to carry it out. This condition is particularly important in the case of less able pupils. Moreover the carrying out of the required tasks facilitates checking of the extent to which the planned outcomes have been achieved, on the part of the pupils themselves, the teachers, and the parents.

In universal secondary education each successive stage is attainable by any pupil, given that he reached a certain level at the previous stage. A minimum range and depth of training must be attained in a prescribed order by all pupils - this general requirement is a

methodological means of implementing the clause in the Constitution of the USSR relating to universal secondary education. The requirements for a guaranteed level of mathematical training which are laid down in the mathematics syllabus must be absolute and their fulfilment is of the first importance.

This predetermines the differentiation in current requirements with regard to the knowledge and skills which pupils must acquire and the differentiation of corresponding planned syllabus must constitute the basic work of the teacher - hence the emphasis in the course of teaching on the most important, the fundamental and the compulsory. On the other hand, if non-compulsory requirements can be ignored, this is a great help in lightening pupils' workload, especially that of the less able. Another way of lessening pupils' work load lies in the distinction between current and final outcomes, the compulsory orientation of the educational process being towards the final outcomes.

The identification of a level of compulsory requirements as regards the formation of mathematical skills, the orientation of academic work towards the unconditional attainment of this level as the basis for subsequent development, changing aims for the achievement of final specific outcomes - all these are part of an important trend of improvement in mathematics teaching which has as its result the all-important development of pupils' practical skills.

The particular nature of mathematics as a science and a method of acquiring knowledge consists in the fact that its technical aspect plays a very important part. The formal mathematical apparatus is a substitute for intellectual effort, but the use of it carries the sphere of influence of intellectual effort to another, more advanced plane. Therefore the most important component in mathematics teaching is the development and "working through" of this apparatus; this is how the practical aspect of education is brought out.

Practical teaching experience shows that a necessary condition for the achievement of this end is the balanced combination of creative and reproductive methods of teaching which are selected on the basis of the pupils' ages and level of mathematical training. In this matter the traditional methodological system comes out more strongly, and for that reason an increase in the role of reproductive methods, including following patterns, represents an important direction in the improvement of the system of mathematics teaching (especially in relation to younger and middle school children and weaker pupils).

The approach to the purposes of mathematics teaching via the development of skills in the solution of mathematical problems, which has been dealt with above, is a unique

projection of the conception which is accepted in Soviet pedagogical psychology: that of the pupil's activity as the basic structural element in the teaching process, and the need for special organisation and direction of pupils' activity in the course of teaching. A consequence of this is changing relationship between propaedeutics and reinforcement, in the direction of the latter. Possibly, before the transition to the new syllabuses the authority of propaedeutics was significantly higher in the eyes of pedagogues. However, experience of the reforms has shown that the effect of timely propaedeutics is insignificant; this is evidently related to the necessarily low level of theoretical generalisation used in this type of work.

The totality of methodological views mentioned, each of which is aimed at improving its own section of the methodological system of school mathematics teaching, has a certain ideological unity arising out of the consciously adopted view according to which a rational combination of new and traditional teaching methods and devices (bearing in mind the conditions and natural boundaries of their use) is an important step in the direction of improving teaching. Perhaps it is no less important to note that this combination must be constantly redefined in relation to new material, new classes, and so on. The ability to discover such a combination is a characteristic of the creative teacher, whose role in the organisation of learning thereby increases significantly. Therefore the centre of gravity of the problem of improving mathematics teaching in school is now moving out of the sphere of mathematics into the sphere of pedagogy, and within pedagogy it is shifting from the area of "the syllabus and the textbook" to the area of "the mathematics teacher". Improvement in the methodological training of teachers is becoming a fundamental factor in the improvement of school mathematics teaching.

In his presentation, the author repeated his view that at the present time the pedagogical aspects of mathematics curriculum development was a more urgent issue than the mathematical aspects; the latter has been settled for the immediate future.

[Editor's note. This is an interesting contrast to his view that in the pre-service training of mathematics teachers, mathematics should take priority over methodology.]

There is a minimum level of mathematical attainment guaranteed to every Soviet school pupil; the problems are how to define this minimum, and how to ensure its attainment. This leads to the question of differentiation of learning. The USSR differentiates by depth of learning - for example by asking questions of different degrees of difficulty on the same mathematical material - rather than by width of coverage as in Britain.

A current issue concerns how to ensure that planned outcomes of teaching actually occur. The necessary outcome of a section of mathematics teaching is the pupil's ability to solve a certain set of problems. "We believe it is impossible to teach without testing". There is a need to find a balance between old and new methods, the "reproductive" and the "productive"; this is a particularly key issue in mathematics.

In elucidation of 'propaedeutics', the author explained that it meant the informal introduction of a concept in preparation for the teaching of it formally later in the school course. While it is of some value, he contended that it is overdone; for example, the idea of function is introduced in Class 1, although it is not formally dealt with until Class 6. It would be better to delay its preparatory introduction until Class 4 or 5.

Session 3. Aims in Mathematics Education

"MATHEMATICS IN EMPLOYMENT 16-18": IMPLICATIONS FOR THE CLASSROOM

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Preamble

In this paper attention is concentrated on the findings of the "Mathematics in Employment 16-18" project (MIE)¹, undertaken for the Cockcroft Committee of Inquiry into the teaching of mathematics in school. MIE was not concerned with those pupils who went on to full-time education in a college or university, but with those who left school at 16 years of age, or within the next two years, to enter employment. Some of these young people would continue their education on a part-time basis following vocational courses at a college. Since completing the study the picture has been affected by the shortage of employment for young people and the trend towards more pupils entering full-time education and vocational preparation beyond 16 years of age. There are several ways in which this might have an effect on the relationship between mathematics in employment and at school, but the rapid development in modern technology and its impact on industrial and commercial practices and on schools is likely to be more significant and is taken into account in this paper.

Introduction

The word "implications" in the title could be misleading, because the findings of the MIE study do not imply a course of action for schools in the same logical sense as when one mathematical statement implies another. What the MIE findings do is to make it possible to compare school practices with the demands that may be made on pupils in their later employment. If in certain respects there is a lack of correspondence, which there is, then whether something should be done to reduce the gap, and if so what, are matters of opinion which will depend on what one believes to be the aims of education, on the

1. "Mathematics in Employment 16-18", a Report. 1981. School of Mathematics, University of Bath.

pressures for change from groups with vested interests and on what is thought to be practicable. This is not the place to discuss these issues at length, but a few words about the first, in relation to mathematics, are necessary in order to keep the matter in perspective.

Aims of Mathematics Education in Schools

It can be said with some confidence that in most countries, not least the UK, mathematics occupies a prominent position in school curricula because there is a general belief that it is useful and that pupils will need a knowledge of mathematics in adult life. Neither of these points can be denied, but what MIE found was that, although the collection of all the mathematics observed was quite substantial, the majority of young people entering employment individually used only a small proportion of the mathematics they had been obliged to study at school, and most requirements were arithmetical. Another study² has shown that they were likely to use even less in their lives outside employment.

There are two common rejoinders to this fact. One is that since generally we do not know which pupils are going to require which pieces of mathematics it is necessary to give all pupils a broad base of mathematics in order to meet this uncertainty. If this view is accepted the nature of that base is still debatable, first in relation to what schools might include and what should be left for specialised vocational education later, and also, of course, how best to present the material which is included in this broad base.

A second rejoinder is that school mathematics is not merely concerned with preparing a base for employment, but that there are also other educational reasons for including mathematics in the curriculum. If this point is accepted, which generally it is, it again remains a matter of opinion as to what the other educational values of mathematics can be, not only in theory, but also in terms of practicability. When values are agreed opinions again differ concerning the best means of realising them. There is no escape from these issues and it will be seen that they underlie the discussion set out below. The nature of the problems is such that if a consensus is ever reached it is unlikely to persist. What actually happens in schools will, in Britain, be the result of the ever-changing interplay of

2. Sewell B. "Use of mathematics by adults in daily life." Advisory Council for Adult and Continuing Education (1981).

such influences as central and local government, individual schools' policies, the qualities of the teachers, examination syllabuses, entry requirements for higher education and professional institutes, expectations of employers and public opinion.

MIE findings

The following features, from among those found in the MIE study, are referred to in this paper, some directly and others incidentally.

- (i) The widespread use of calculators and the beginnings of the micro-computer revolution.
- (ii) The continuing decline in the need to manipulate fractions of the form $\frac{a}{b}$.
- (iii) The appearance in work situations of idiosyncratic methods of calculation.
- (iv) The almost total absence of algebraic manipulation, but widespread need to be able to substitute values in formulae.
- (v) The limited range of geometry and trigonometry, mostly associated with technical drawings.
- (vi) The obvious fact that all the mathematics was embedded within concrete contexts, frequently associated with counting and measuring.
- (vii) Approximation and estimation, when they were required, being related to physical constraints rather than being determined by mathematical considerations.
- (viii) Employees soon developing facility in the mathematics they actually used.
- (ix) The widespread lack of appreciation of mathematics among young people and adults as anything other than a possibly useful tool.

Calculators and Micro-Computers

The widespread adoption of the use of electronic calculators in commerce and industry has led to several important trends. First there has been a massive decline in the use of slide rules and mathematical tables such as logarithms, trigonometric functions, powers and

roots. It is now difficult to see that there is any convincing reason why teachers should continue endeavouring to make pupils proficient in the use of these out-moded devices. In time they will take their place alongside Napier's Bones. Moreover, the introduction of scientific calculators into schools should release much time previously spent introducing and ensuring the proper use of other calculating aids, which could now be directed towards other purposes, such as developing a better understanding of the mathematical functions themselves.

Secondly, because calculators are now inexpensive and portable, compared with the earlier generation of desk-top models, the use of them for the four basic arithmetical operations and percentages has increased enormously. They are used by many people for all but the simpler calculations, and in particular the longer pen and paper algorithms for multiplication and division are rarely going to be used by anyone.

Much time has been spent teaching these algorithms and endeavouring to find better ways of doing so, but with limited success for many pupils, especially when working with decimals. We again have to consider whether there is any value in continuing to teach these methods and in pupils practising them to achieve competence. Because the methods will not be used when pupils enter employment, facility in them will atrophy rapidly, unless even more time is spent reinforcing them. The question is sometimes asked: how will an employee perform a long multiplication if the calculator breaks down? But the cost in terms of time and effort of ensuring that the employee can readily recall a pen and paper method would seem to be out of all proportion compared with ensuring that a replacement calculator is always available.

We can go further. There are pupils who have left school still not confident in their use of algorithms for simpler processes such as subtraction. In the past these pupils have been at a serious disadvantage in employment, but the calculator is now a means of liberating them from the bondage of these inadequacies and enabling them to perform tasks otherwise beyond them. This extension of capability can operate at all levels; for example a young craftsman who avoided situations requiring trigonometry, because he was confused about the negative characteristics in logarithms, lost his fear after he had acquired and taught himself to use a scientific calculator.

Idiosyncratic Methods and Practising Routine Procedures

MIE found that in some situations where calculators were not available, rather than using methods taught at school, employees adopted idiosyncratic methods they had been shown by another employee or devised themselves and which were often peculiarly appropriate

for the job in hand. Such adaptability would seem to have many advantages and to be worth encouraging, but it clearly requires a degree of confidence in one's mathematical ability, albeit at an elementary level, which so many people appear to lack.

If instead of expending so much effort teaching standard algorithms, even for addition and subtraction, teachers directed their efforts towards ensuring pupils understood the fundamental nature of the elements and operations, together with building confidence in using a calculator and in devising ad hoc methods, this might be a better use of the available time. It is possible, although by no means certain, that pupils sometimes gain something in the understanding of a concept by repeated practice of an algorithm, or by repeated application of the algorithm to problems, but it is also possible that with calculators already readily available, and micro-computers increasingly so, other more effective ways of deepening understanding can be developed. Additionally, pupils at all levels could be encouraged to exploit the power of the calculator and micro-computer to solve problems which they would otherwise find intractable.

The above argument can be generalised to cover many aspects of the teaching of mathematics. We can assume that the production of and access to software will become better organised, so that if any particular known procedure is required an appropriate program can be called up to handle it. For example, the drawing of pie-charts to represent statistical data has found a place in the syllabuses of most English schools. Except in the simplest cases when the total number of elements is directly related to 360, many pupils have had difficulty calculating the required angles. The process is made easier with a calculator, but now there is available to schools a computer program, which on being fed the data will produce the angles and the diagrams. This being so the habit of making pupils practise the process in order to develop competence, which has always been of doubtful value, is now seriously called in question. Very few pupils will ever be called upon to draw a pie-chart in their employment and even if the company does not have a program of its own it may not be all that long before a national system of software libraries is developed so that a suitable program can readily be called up for the task if required. Otherwise the only demands on pupils may be to be able to understand pie-charts when they see them used on television, in the newspapers or in other documents. But again, practising a method of construction may not be the most effective way of developing this understanding. The same kind of argument could be raised in relation to a wide range of topics currently found in school mathematics syllabuses; the argument is readily accepted at higher levels, as instanced by advanced university students who require the ability to interpret and perhaps use such statistical procedures as the analysis of co-variance. How many of those students are expected to labour over exercises with pen, paper and perhaps

a calculator, rather than employing a computer package? Are we to suppose it must be different for children and less sophisticated concepts? Teachers may be afraid to give up the classroom methods that have been widely employed for many decades, even though evidence has shown these methods to be of limited effectiveness, but modern technology is providing compelling pressures for a rethinking of strategies.

Fractions ($\frac{a}{b}$ with $a, b \in \mathbb{Z}, b \neq 0$)

What has been said concerning the implications of calculators of course has relevance here, only with added point and raising other general issues. There are available calculators and computer programs that will perform the four basic operations with fractions, but in relation to employment the need for pupils even to understand the general rules, let alone be able to perform them competently, is now called in question. The MIE study has shown that in the UK the need to manipulate fractions is almost confined to the engineering industry, where addition and subtraction may be required for fractions of inches with denominators powers of 2 up to 2^6 . This restricted set makes the processes much simpler, no new denominators having to be found. Apart from engineering some other situations involve a sub-set of the same fractions, and occasionally another fraction, such as a third or a number of fifths, may appear. However in these latter cases, and increasingly in engineering, if calculations have to be done the fractions are usually converted to decimals and a calculator is used. In engineering, fractions also appear in calculations of gear trains involving facility with multiplication and a good grasp of equivalence. However the gradual change to metric units and the increasing use of computerised machine-tools with digital meters and machines with automatic gear boxes are bringing about a drastic reduction in the need to manipulate numerical fractions.

However, what the implications are for schools is not so clear. Even recently published text-books in Britain still contain pages of practice in the four rules with fractions, unsupported by any genuine applications; but this tradition is looking increasingly anachronistic. Is it possible that there could be a massive reduction in the amount of work devoted to mastering fractions without any serious harm being caused to the education of the pupils? In terms of utility alone familiarity with the concept of a fraction as a number of parts, so that pupils would know, for example, what "three sevenths" and $\frac{3}{7}$ meant, together with the notion of equivalence and how to combine fractions applied to halves and quarters, fifths and tenths and other similar pairs, together with tenths, hundredths etc. for the decimal system, would probably be enough for most pupils. However there are several broader issues to be considered.

First, consider the algebra of the real numbers. In Britain, although many teachers are always looking for better ways of teaching algebra, there seems to be a general acceptance that many pupils will not get very far with the subject during the years of compulsory schooling. From the point of view of performing satisfactorily in employment this is unlikely to be a disadvantage, MIE having found that the need for young employees to perform algebraic manipulation was rare. Formulae were encountered in many different situations, but all that was required was substitution of numerical values, with the fraction notation being accepted simply as an instruction to divide. Some employees had to draw and interpret graphs of data, but not graphs of mathematical functions.

Nevertheless, for other reasons it may be deemed educationally desirable for all pupils to study algebra as far as they are capable of doing so, and of course many will go on to higher studies which depend on algebra. But one cannot get very far in algebra without becoming involved in the manipulation of algebraic fractions and it seems likely that fluency here would depend on a sound familiarity with the general rules for numerical fractions. At present these general rules are taught long before they are required in algebra and without any really convincing applications being available for motivation. It might be more effective to postpone the introduction of the rules until shortly before they are required in algebra, which might provide some motivation within mathematics itself.

Introducing the general rules for fractions much later than is customary could have several other advantages. It would mean that those pupils whose progress in mathematics is such that they do not meet algebraic fractions at school, will not need to be burdened with the general rules for fractions. Other pupils would meet the rules when they are more mature and perhaps better able to understand the logic behind them. Moreover, given more time it may be possible for teachers to prepare the ground more effectively beforehand. A stepping stone to a full treatment of the rules could be elementary work on the combination of probabilities, which might arouse motivation through concrete applications and opportunities for practical work.

A second broader issue is the extent to which pupils should be introduced to the nature and construction of the number systems, natural, integer, rational, real and complex, as part of their personal education, irrespective of whether much of this knowledge is put to use in employment or higher studies. MIE not only found that manipulation of fractions was diminishing, but also that real numbers, as such, were not in evidence. Square roots, trigonometric ratios and π appeared in technical situations, but could be handled with sufficient understanding in practice as rational approximations, without any deep appreciation of the concept of irrationality, still less incommensurability. Applications of complex numbers, like many other topics, only appeared at much higher levels, for example in electrical circuit design, and would be covered in post-school studies.

For most pupils the mastery of the number systems involving much repetitive practice and application is unlikely to be required in employment or higher studies and the broader educational aims might be satisfied by some acquaintance with the systems and their logical interdependence. In Britain although the logical structure is presented to mathematics students in sixth forms (16-18 years of age) only a little is attempted prior to this. The abstract nature of the topic poses daunting pedagogical problems, but if teachers were released from having to drill pupils in the rules and could postpone the work until later it is possible that something could be achieved at least up to the rational numbers for many pupils, and beyond this for some.

Concrete Contexts

All of the mathematics MIE found, including the geometry and trigonometry of the right-angled triangle which was one of the more advanced topics, was related to some concrete context often involving counting and measuring: numbers of goods, dimensions of objects, quantities of chemicals, gas flow, monetary exchange rates, etc. Working in these contexts it appears that young employees can quickly develop confidence and competence in the required calculations, which are often being performed every day and many times a day.

Moreover, by experience they can become aware of the magnitudes of quantities and measurements that occur and of the degree of accuracy that is acceptable in calculations, so that a sense of appropriate estimations and approximations is developed through practical involvement, rather than by mathematical considerations alone.

Perhaps there are lessons for schools here. It is possible that there could be considerable advantages for many pupils, especially those not inclined towards higher studies, in relating the teaching of mathematics more closely to practical activities. Not only could this give more purpose to mathematical calculations, and perhaps develop keener awareness of whether answers were sensible, but it might also provide a base for the development of the skills of estimation and approximation, which when approached abstractly pupils seem to find so difficult.

Attitudes to Mathematics

MIE found that it was very rare for either a young person or a more mature adult to see mathematics as anything other than a possibly useful tool. Some young people could not even see that; "What's the use of algebra?" was the kind of comment heard on numerous

occasions, and Sewell's study² showed that many adults shied away from becoming involved in applying mathematics of even a most elementary kind to everyday situations. There appeared to be little appreciation of mathematics as the most logical of subjects, as the foundation of much science and technology, as a human achievement in its own right to be compared with and perhaps surpassing other arts, as an activity through which personal satisfaction could be gained, or as a way of thinking of value to the individual. This appreciation is the essence of the broader educational aims referred to at the beginning of this paper, but if it is believed to be of value a major reform in the ways pupils are brought into contact with mathematics will be necessary to cultivate it.

It is possible that imposing on pupils the study of concepts beyond their level of maturity and largely detached from the real world, and resorting to repetitive practice in order to achieve a semblance of progress, have been at the root of the negative attitudes towards mathematics among young people and adults. There now exist strong arguments for a radical rethinking about what goes on in the mathematics classroom. The suggestions in this paper are intended to have a bearing on this, but clearly there is a need for much more hard thinking and diverse experimentation if an appropriate pedagogy is to emerge.

Mr Fitzgerald's paper led to one of the most interesting discussions of the Seminar, summarised below under the main themes touched upon.

Clarification of the research procedure

In response to questions it was made clear that the research involved meeting young people at their place of work, and initially to study what they did as work, irrespective of mathematics; the observation was then focussed down to the mathematical aspects, and the particular mathematical procedures used were identified through discussion with the young worker, monitored through the researcher's own experience, and checked with the supervisor - though it was often found that the supervisor's perceptions did not completely coincide with what had been observed. The work was then examined in more detail within the research team, a report sent back to the employer for verification, and finally passed on to the Cockcroft Committee.

About 90 establishments were visited by the 3-man research team, largely from manufacturing industry (another project was looking more closely at other forms of employment) but covering a wide range of activities within this - operatives, skilled craft trainees, laboratory technicians, clerical staff, etc. There was no deliberate selection in

terms of intellectual ability, but the fact that all these young people had chosen to leave school at age 16 implied that they did not include many of the most able pupils.

Several questions were raised during the discussion about the extent of the available research. Were predictions of future needs taken into account? Mathematics is not only needed for work, but also in ordinary life, eg in bringing up a family; did the team look at this? It was made clear that this research was not seen as the last word on the subject, but that these factors had not been forgotten; in particular, there had been a parallel study on the mathematics used in adult life (reference 2 in the paper). The most significant feature about this latter study is that over half of the original sample of adults refused to be interviewed about their use of mathematics. The research revealed a strong antipathy to the subject, and a tendency to avoid using it wherever possible. Superficially there appeared to be some correspondence and some anomalies between the two research studies.

School mathematics and the expectations of society

It was suggested that society makes certain requirements about mathematical education, which are implemented by schools; to what extent do these correspond to the mathematics needed in work? The evidence was that all the mathematics the young people needed was already in existing school syllabuses, but that this could be misleading since what appears by name in a syllabus may be used differently in a work context. For example, different methods might be expected in the two situations: at school, children might be required to multiply a 2-digit number by a 3-digit number using pencil-and-paper, but on the job this would invariably be done with a calculator. Society (and this term embraced factory supervisors) was always conservative in its attitudes: "we like them to be able to do it properly". But it seems likely that in time new technology will become widely accepted, and will lead us to abandon teaching some of the traditional algorithms - as has already occurred with the pencil-and-paper method of extracting square roots. We cannot expect public opinion to move too fast.

One problem is that other mathematics is still needed by pupils who will go on to the university. We need to find a way of organising teaching so that the majority whose use of the subject does not go that far are taught just the practical skills they will need.

Implications for the school syllabus: the identification of aims

The question remains, if certain aspects of mathematics are not necessary for work, should they nevertheless be kept in the syllabus, and if so why? In our concern to improve

the methods of teaching we too often concentrate on "how" and "what", and neglect "why".

The point was put that the study of mathematics also develops cognitive abilities, and inculcates habits of hard work. However we do not have convincing evidence of the former - and the latter could even be said to be contrary to the ethos of mathematics, a subject in which by hard thinking it is possible to avoid hard work. It was asserted that even young people who go directly into work need some personal mathematical development, which might not occur if machine calculators were emphasised at the expense of more conventional teaching; the counter-argument that, by reducing laborious work on calculation, there were better opportunities for developing mathematical thinking, was supported by some research in English primary schools, but some participants dissented from this view.

The discussion was then widened. Investigations such as MIE are very necessary, but the most important thing is the interpretation of the results. Although the requirements of work are necessary, there are also other considerations. For example, although transformation of formulae is not used in work situations, it is important at the beginning of calculus; so if we decide that calculus is a desirable element in school mathematics, we must continue to teach these supporting skills. It is also important to appreciate that in the work environment mathematics is used under pressures imposed by simultaneous physical activity; it could be dangerous to equate what a person should be taught in school with what he should be taught as a worker. If we were to start teaching fractions later (for example), this might have unfavourable consequences in the overall development of the curriculum; were any studies made along these lines? It was agreed that this could be a problem, but we ought to keep an open mind on such questions until specific research evidence was available.

Speaker's summing-up

The most important finding from the research is the attitude of young people towards mathematics, and their perceptions of school mathematics. We as educators enumerate a variety of aims of mathematical education: development of logical thinking, appreciation of mathematics as part of human culture, etc. But when this question was discussed with the young people at work, it was found that they either saw no purpose in school mathematics, or they thought that they had been taught mathematics because it was useful. Even if they didn't use it themselves, "it must be useful, or we wouldn't have been taught it".

It is not the job of the school to prepare pupils in detail for the mathematics they will ultimately use at work; but the existence of these attitudes does present us with some challenging questions about our aims in education.

THE AIMS OF MATHEMATICS EDUCATION

Dr Geoffrey Howson, University of Southampton

1. Some aims

Four centuries ago, in 1581, Richard Mulcaster, Headmaster of a London school, set out the claims of mathematics for a place in the school curriculum:

'Such studies require concentration, and demand a type of mind that does not seek to make public display until after mature contemplation in solitude ... The Mathematical Sciences show themselves in many professions and trades ... whereby it is well seen that they are really profitable; they do not make outward show, but our daily life benefits greatly by them ... Mathematics are the first rudiments for young children, and the sure means of direction for all skilled workmen, who without such knowledge can only go by rote, but with it might reach genuine skill ... In the manner of their teaching (they) also plant in the mind of the learner a habit of resisting the influence of bare probabilities, of refusing to believe in light conjectures, of being moved only with infallible demonstrations.'

In 1982 the Cockcroft Committee gave its view on why one should teach mathematics. (Here, it should be noted that in England there are no centrally laid down 'national' aims for mathematics teaching in schools, hence the need for these to be specifically considered by the Cockcroft Committee.) It, like Mulcaster, stressed the 'usefulness' of the subject, but emphasised that 'usefulness' can be perceived in a number of ways: 'for many it is seen in terms of the arithmetic skills which are needed for use at home or in the office or workshop; some see mathematics as the basis of scientific development and modern technology; some emphasise the increasing use of mathematical techniques as a management tool in commerce and industry. We believe that all these perceptions ... arise from the fact that mathematics provides a means of communication which is powerful, concise and unambiguous.' The importance and usefulness of mathematics in other disciplines was advanced as a justification for teaching the subject and it was also claimed that correctly taught mathematics could develop 'logical thinking, accuracy and spatial awareness.'

In fact, these modern aims overlap considerably those set out by Mulcaster, ie mathematics could be seen as:

1. contributing to the basic knowledge of any educated citizen,

2. contributing to the study and advancement of numerous disciplines, professions and trades,
3. contributing to a student's general education through the inculcation of particular attitudes or approaches.

Cockcroft was, however, to add a fourth reason for teaching mathematics, namely:

4. The inherent interest and appeal which it can have for many children and adults.

This last aim, 'giving insight and confidence, deepening feelings, engendering delight' (Mathematics in Secondary Modern Schools. Mathematical Association, 1959), is indeed a fairly late addition to the list of aims of teaching mathematics. Moreover, as the Mathematical Association report pointed out, it was an aim that had received little attention in the education of the average child as opposed to the more able mathematical student.

These four aims are by no means exhaustive and I believe they omit an important element, namely:

5. An appreciation of what mathematics contributes to society.

The aims I have quoted refer to the students' acquisition of worthwhile skills, techniques, practices and working knowledge: the emphasis is on having the student 'perform', rather than 'appreciate' or 'be aware of'. (Consider the analogy with music - performing and appreciating.) In the opening years of this century Godfrey, then headmaster of a naval college at which the future King George VI was a pupil, wrote that in the schools there was a need to 'train up an educated public opinion which can appreciate the work of mathematicians and so form a social atmosphere in which mathematical study can live'.

In the ensuing years this 'appreciation' aspect has been given very little attention, and I should like to add it to my list of aims as a fifth component.

2. Some viewpoints

The quotations in section 1 all come from the educational community. They seek to justify the teaching of mathematics (thus, whether consciously intentional or not, the relevant chapter in the Cockcroft report is entitled 'Why teach mathematics?' and not 'Why learn

mathematics?'). It would be somewhat naive, however, to assume that these views concerning the aims of mathematics education were accepted by other interested parties. It is, I believe, of value then to consider possible tensions which arise because of these differences.

If one wishes to convince a government of the need to give financial support to mathematics education then it is on aims 1 and 2 that one places the emphasis. Thus, for example, Wirsup's The Soviet Challenge would appear to be trying to rouse the US Government into action by claiming that aim 2 is being more successfully realised in the USSR than in the US. Wirsup stresses the fact that relatively few US students study calculus. Yet being able to integrate $\cos 5x$, say, can hardly be classed as basic knowledge for the performance of responsible citizenship, and integration by substitution is not the most obvious topic for realising aims 3-5. Clearly, the fear that Wirsup is seeking to create is that a widespread knowledge of integration amongst a country's populace is likely to create a climate of thought that will accelerate that nation's technological and scientific advance. In a similar way an employer, if he has rationalised that far, wants a numerate workforce because it will further the work of his firm.

Even within the educational community aims 1 and 2 - with 'student' substituted for 'citizen' in 1 - are those most frequently stressed by, for example, science teachers and university engineering departments. Here, though, the emphasis switches from the student being 'educated' to his being 'trained'. Thus in the English system the 16-18 curriculum (for those students expected to progress to higher education) is very much influenced (some would say, almost totally controlled) by the wishes of universities and polytechnics and these very much emphasise the acquisition by students of well-defined areas of content and of specific techniques. Put crudely, there is a dichotomy between school mathematics pre- and post- the age of 16. The former is seen as something to be determined through consideration of the aims of many sectors of society and in a general educational context, the latter as a preparation for higher education. In recent years, as disciplines within higher education have become more and more quantitative, there have been increasing and differing demands on 16-18 school mathematics. In particular, the increasing need for statistics has meant that less emphasis has come to be placed on mechanics, a move opposed by engineers, physicists and some other.

In addition to considering the aims of mathematics education as perceived by governments and other finance-producing authorities, it is also essential to consider the views taken by parents and, most importantly, pupils.

'Many of the children's parents saw education in general, and mathematics in particular, as a set of facts to be learnt and a collection of procedural solutions to practical computational problems: as difficult routines which children must be made to reach. When they saw their children doing something which looked like play they did not conceptualise it in the same way as the teachers, instead they complained because their children were not working and didn't seem to be learning anything. The teachers' frustration lay in their difficulty in making the parents see it differently ...' (Corran, G. and Walkerdine, V. The Practice of Reason, London University Institute of Education, 1981).

Yet, of course, parents can never view mathematics education and its aims in full generality: they are bound to consider the aims of their children's mathematical education, and naturally enough the prime aim will almost certainly be seen as the acquisition of certificates which may well serve as passports to worthwhile jobs. There is a divergence of aims which must be bridged if the 'frustration' of the teachers is to be eased.

How do our pupils view the aims of mathematics education? 'I haven't heard anyone, anyone except a mathematics teacher, say that math is great. You gotta know how to figure your income tax, how to get money from banks, how to buy and sell stuff, know the stock market and maybe a couple other things and that is it' were the views of one budding capitalist as reported by J T Fey ('Mathematics Teaching Today: Perspectives from Three National Surveys', Maths. Teacher, 72 (1979), 490-504). These U.S. surveys found that teachers had difficulty in justifying their teaching of mathematics and when pressed fell back on the 'familiar mental discipline argument' (teachers of other subjects would presumably have experienced similar or even worse difficulties): 'the students seem to sense this same underlying objective in studying mathematics, though they express somewhat more cynical regard for the goal'. I hope (without the benefit of research findings) that English teachers would have less difficulty in justifying their work. I fear, however, that students would be equally cynical if told that their mathematics lessons were serving any great educational purpose in the sense of our 3rd aim.

The indications are that English students, like their US counterparts, feel that they need mathematics as future citizens 'even though they have found no environmental application for what they were studying'. Mathematics must be useful, otherwise why would everyone make so much of a fuss about it? There does seem, however, to be a dearth of information concerning what pupils and society at large see as the aims of mathematics teaching in schools.

Steig Mellin-Olsen has attempted (*Ed. Stud. Maths*, 12 (1981), 351-67) to provide a theoretical framework within which we can study the students' rationale for school learning. He distinguishes between two possible rationales. Thus the S-rationale is evoked in the pupil by a 'synthesis of his self-concept, his cognition of school and schooling, and his concept of what is significant knowledge and of possible future value'. The key is that for some reason or other the pupil sees what he is being taught as significant for himself, according to his norms. Since such norms will be much affected by, say, geographical differences or those of social standing, they will tend to fall within the field of the sociology of education. The I-rationale is related to the way in which the school is viewed as 'instrumental' in providing the pupil with a 'future'. Its most important manifestation is in the way in which the external examination can provide certification. Thus, the external examination system can provide I-rationale for many students -and, as we have noted, parents. This I-rationale will exist almost independently of the actual content of the mathematics examination: the mere existence of an examination called 'mathematics' which is accorded weight and is used as a sieve towards further advancement is sufficient to generate I-rationale. Mellin-Olsen argues that the major task of the didactician and teacher is to ensure the greatest overlap of S- and I-rationale, ie to ensure that the two innate forces which motivate the learner should reinforce each other.

Whether or not one agrees with Mellin-Olsen's analysis of students' rationales for learning mathematics, it would still seem to be obvious that politicians, employers, teachers, parents and students will all have their own rationales and that where these conflict there will be problems. This is not to say that conflict can or should be avoided. What is essential is to recognise that different groups will have different rationales and it is the educator's job (but who is the educator: teacher, administrator, text-book writer, ...?) to recognise this, to mediate and then to make all the interested parties aware of the rationale on which the school course is founded.

3. Aims in practice: content links

Since mathematics teachers and curriculum planners everywhere are presumably guided by some particular set of aims concerning mathematics education, it would be reasonable to expect that a comparative study of curricula would be useful in illustrating differences in aims and in the ways that these are weighted. Considerable data concerning curricula are currently being generated by the Second Study of Mathematics mounted by the International Association for the Evaluation of Educational Achievement. The results, are, to say the least, confusing!

How is one to account for the apparent emphasis which Chile and the USA, say, place on logic (for 16-18 year olds), whereas neither stresses statistics? Does this mean that these countries still lay great stress on aim 3 and rather less on aims 1 and 2? Why should Hungary appear to lay greater emphasis than most countries on geometry, and less on algebra? Are these differences evidence of the practical interpretation of differing aims or the lingering on of different traditions and responses to different pressures exerted by universities? ('Pressures', of course, are the evidence of other people's aims!)

It is in geometry that the most significant differences of content appear, and here differences in aims are apparent. The English approach of recent years is consonant with aims 1-4, yet it is obvious that the attitudes and approaches it seeks to inculcate - inquiry, experiment, visualisation - are not those traditionally sought in geometry teaching (or even those sought by French, Belgian and some US reformers), ie deductive reasoning, an axiomatic approach. It could be said that the French and Belgians had a sixth aim:

6. to acquaint the learner with the methods and ways of thought of practising pure mathematicians.

(Here, it is perhaps useful to disentangle the aim from the way in which it was interpreted in the 1960s and 1970s when the mathematical community misjudged the apparent power of structural and axiomatic methods - one cannot believe any mathematician in the 1980s would press the claims of teaching categories in the secondary school - or even of teaching affine geometry before Euclidean. In Britain it was probably the varying emphasis placed on aim 6 which most clearly served to distinguish between the projects of the 1960s. Thus the Swansea Project (concerned with pure mathematics teaching in the age range 16-18) placed considerable emphasis on that aim, and the Midlands Mathematical Experiment accorded it greater weight than did, say, the SMP, particularly in its treatment of algebra and geometry. In general, though, the projects had largely similar aims; it was in the translation of aims into curricular detail that differences arose.)

Content is but one way in which aims are translated into practice. The method of teaching used should also reflect the aims of the teacher, and presumably that of the system in which he works. Again, however, it is not obvious how the relative emphasis, say, on child-centred education within different countries correlates with accepted national goals for education (and, more specifically, how the general educational ethos is represented in mathematics lessons - in some schools which have attempted an informal, integrated (multidisciplinary) approach to the curriculum in general, mathematics has still been

taught, more formally, in separate time-tabled slots). Often, too, general educational aims are separated from the aims of teaching within mathematics classes. For example, it must be accepted that a general aim of education is to foster co-operative work and effort, yet within mathematics lessons the emphasis is still mainly on individual work and competition. (Witness, also, the emphasis on olympiads as individual tournaments: why not have a team session in which students can collaborate to solve problems?)

4. How does one change aims?

It is relatively easy for a curriculum designer, or textbook writer to adjust his aims, to decide to give a greater emphasis to certain features of the curriculum, to introduce new teaching methods, etc. Yet, if those changes of emphasis are not explicitly communicated to, and comprehended by, the classroom teacher, it is unlikely that the intended developments will be successfully accomplished.

The teacher has a body of aims (which frequently he would find difficult to make explicit) and a concept (perhaps peculiar to himself) of what it is to be a teacher, what a teacher should be and do, and what position he should hold in relation to his students. It is clear that in the developments of the past 20 years insufficient attention has been paid to the difficulties of persuading teachers to adjust their aims of teaching and their self-concept of how a teacher behaves.

Teachers have been introduced to new content, new mathematical ideas and techniques with comparative ease. They now 'know' about matrices, transformation geometry, etc and those teachers who delighted in teaching quadratic equations are now equally delighted to teach operations on 2×2 matrices. The second aim of curriculum development, to persuade teachers to reconsider and modify their teaching methods, has proved more difficult to effect. Here the subjective view of what constitutes good teaching (of necessity) looms large. Unhappy experience with new methods can soon cause a reversal to the old, even before the expertise needed to give the new methods a fair trial has been gained. Important psychological barriers have to be overcome if permanent changes in teaching method are to be effected. Finally, there is a need to persuade teachers to reconsider their aims in teaching mathematics. This would seem the most difficult task to achieve. The teacher who switches happily from quadratic equations to 2×2 matrices takes his previous aim - the passing on of techniques which can be usefully applied in examinations - with him. An encouraging feature, however, is that once a teacher can be persuaded to rethink his aims, then such a change would usually appear to be lasting.

It is not sufficient, then, to consider what aims we (as a small, particular group of educators) have for mathematics education. These aims must be checked against those of other groups of concerned people to see where clashes are likely to arise. Following that must come a process of mediation and after that, if successful curriculum development is to be effected, come the tasks of explaining decisions to others and convincing them of their value. This is a vital part of in-service education, a process which involves more than re-training teachers.

APPENDIX

The Teaching of Geometry

In order to exemplify the problems of converting general aims into specific curricular ones, I have chosen certain quotations and classifications within the general field of geometry teaching.

(A) The first quotation illustrates the problem of deciding to what extent general educational aims and our understanding of developmental psychology should over-ride scientific and purely mathematical ones. "In schools, where the main objective must be a harmonic cultivation of a pupil's every talents the purpose would be definitely missed if analytic geometry was to dominate mathematics tuition. A pupil on his lowest level of education clings only to the tangible and the school's objective must be slowly to draw him away from that, to 'elevate' him from the particular to the general... If, however, not the education of a human being, but rather science itself is the objective of geometrical activity ... then analytic geometry undoubtedly attains predominance ... with generality and as quick as lightning it ... often achieves in an atom of time what a study of the geometry of the elders cannot achieve in a thousand years". (Martin Ohm, 1819).

(B) The second concerns a constantly recurring problem, balancing 'meaning' and 'understanding' against the acquisition of powerful techniques whose workings may not be fully appreciated. The central problem here is that the classroom teacher always has an aim which is rarely given much attention by educators outside the classroom:

7A To find some mathematics which the pupils can do.

(It is in an attempt to satisfy this aim that many curricular innovations have been trivialised by teachers, eg work on matrices. The conflict between this aim and other more 'educational' ones can be considerable).

Pupils, too, have the aim:

7B To obtain correct answers (and ticks) for their mathematics problems.

Again, there is a need to recognise that this aim exists and to provide ways for its attainment which are not in conflict with other aims.)

'It will be evident that there is a marked distinction between Geometrical and Algebraical reasoning. To comprehend the one, the whole process must be kept in view from the commencement to the conclusion -while in Algebraical reasonings ... the attention is altogether withdrawn from the things signified, and confined to the symbols, with the performance of certain mechanical operations, according to rules of which the rationale may or may not be comprehended by the student'. (R. Potts, 1845).

(The French mathematician, René Thom, was to make a similar point when speaking at the 1972 ICME. There he described geometry as rich in meaning, weak in syntax, whereas algebra was rich in syntax, weak in meaning. The extent to which we try to develop meaning, rather than take refuge in that mechanical syntax which Potts feared, is still a vital issue in the planning of curricula).

(C) Here we give a list of objectives for teaching geometry as set out by Wynne-Willson in Geometry (Blackie, 1977) together with a summary showing the contrasting ways in which various British projects have chosen to stress these. (The number of asterisks indicates the emphasis placed on each goal. The prefix G indicates a 'geometric objective', M a 'mathematical objective'.)

- G1. The ability to draw, sketch and use geometrical instruments.
- G2. Knowledge and understanding of the properties of geometric figures, both in the plane and in space.
- G3. Knowledge and understanding of the properties of geometric transformations.
- G4. Awareness of the links between geometry and the real world.
- G5. Pleasure in shape and form.
- M1. Geometry as an introduction to imaginative thinking.
- M2. Geometry as an introduction to deductive method.
- M3. Awareness of the links between geometry and other parts of mathematics.
- M4. Pleasure in mathematical ideas.

OBJECTIVES

	G1	G2	G3	G4	G5	M1	M2	M3	M4
Durell	***	***		***	*	*	****	*	***
SMP A-H	****	*	*****	**	*****	***	*	*****	*
SMP A-G, X-Z	****	**	*****	**	*****	***	**	*****	*
MME	**	***	**	***	**	***	***	****	***
MMS (first edition)	*	***	***	**	*	**	****	**	****
Maths for the Majority	***	**	**	*****	*	****		**	*

(Durell is a 'traditional', 1930s textbook; the others were published by projects in the 1960s).

Note the need to match the specific objectives listed here with the general aims for mathematics education set out earlier.

In presenting his paper, Dr Howson remarked that aims need to be translated into curriculum. It is important to keep in mind the distinction between the intended curriculum (by planners), the implemented curriculum (by teachers) and the attained curriculum (by pupils). It is necessary to recognise that various groups - teachers, parents, higher education, etc. - have different aims in respect of school mathematics.

There are differences in attitude in the interpretation of aim 3, the contribution of mathematics to general education; this aim appears to be underplayed in Britain. Aim 5, relating to the appreciation of mathematics, is frequently ignored.

Logical reasoning and axiomatic method

Much of the discussion centred round aim 6 - understanding the ways of thought of pure mathematicians - narrowed down to logical reasoning and axiomatic method.

A distinction was drawn between local reasoning, which was generally considered desirable, and a fully axiomatic approach. One big question is to decide whether to introduce all the axioms at the beginning, or to introduce them one-by-one when needed. The latter can be confusing to a child, who could get the impression that you introduce an axiom when you reach an impasse. Although this is what happens in pure mathematics, at that level one then recycles - which is not practicable at school, where one of the major difficulties is arriving at the axioms. Even at university we often teach axiomatically without explaining how the axioms developed.

It was pointed out that by introducing axioms one was not necessarily trying to teach axiomatic method; but it did give pupils a basis from which to learn about proof. This was a more realistic target for pupils of age 13. However, it was suggested that geometry may not be the best vehicle for doing this, not even algebra; it could be done with simple arithmetic results, such as "the sum of two odd numbers is even", which could be proved either using very elementary algebra or with the aid of a figure.

To the question whether an axiomatic approach contributed to any of the other listed aims, it was suggested that it could contribute to general education (aim 3) and has an inherent appeal for some children (aim 4); but this appeal was thought to be limited to a very small number of pupils.

Implementation of other aims

It was suggested that the aims should be expanded to make clear that we are not just concerned with mathematics for mathematicians. There might be another aim, "onto-didactics", to acquaint pupils working in science with the activities of mathematicians. If we formulate objectives from this point of view, the axiomatic method will occupy a less prominent place.

The aims also require extension to take account of the general educational value of mathematics - for example, to refer to the development of moral qualities. However, what is most important is to find out how to translate general aims into concrete form; this is the way in which we can see how the aims have been achieved. The most effective way of doing this is not to try to split aim 1 (education for citizenship) into a lot of concrete aims, but to accumulate concrete objectives and then correlate them with the aims as expressed.

Speaker's summing up

Kropotkin wrote of the course in philosophy which he received as pulling together different strands from various parts of his education. There might be a role for mathematics in this context in relation to science - we recall that what we now call applied mathematics used in England to be called "natural philosophy". But this would require a revolutionary change in our approach to these questions.

Finally, Dr Howson set the Soviet participants some homework: to assess the objectives of Academician Pogorelov's new Geometry book for Classes 6-10 by awarding it asterisks according to the table at the end of his paper. This challenge was subsequently taken up by a number of the participants.

Session 4. Achieving a Balance in Mathematics Education

ON THE RELATIONSHIP BETWEEN CONTENT AND FORM IN SCHOOL MATHEMATICS

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Throughout the world over the last ten years the teaching of mathematics in school has undergone significant changes, the purpose of which has been to bring the school course into line with the modern state of mathematical science. In a number of countries elements of set theory and mathematical logic, the basic concepts of the theory of algebraic structures, geometric transformations, the beginnings of vector algebra and basic differential and integral calculus have become part of the school syllabus.

Radical changes in the content of mathematics as an academic subject could not fail to affect a second important aspect of mathematics teaching in schools - the new content led to new conceptions of the style of presentation, also aimed at a greater similarity to the style of modern mathematical science. Many features of traditional teaching were deemed old fashioned and not in accordance with the new level of rigour required by the new content, and began to be seen as contradictory to those demands that mathematical science makes upon mathematics teaching.

This practical universal drive towards rigour is reflected in one of the most characteristic features of the new style of presentation of the school course - the special emphasis on the strict definition of mathematical concepts covered by the course, "definitional formalism", or more precisely, "definitional rigorism". A firm basis for introducing this logical emphasis into the methodics of teaching was provided by the structuring of the school course around set theory - a perfectly natural reaction to corresponding developments in mathematical science itself.

The set theory approach, enriched with the simplest concepts of mathematical logic, made it possible to give precise formal definitions to many of the concepts in the school course, the treatment of which in traditional presentation had seemed insufficiently clear, insufficiently rigorous, and now in the light of the new tendency sometimes appeared totally unacceptable. Among the concepts of this sort were some of the fundamental, central concepts of the school mathematics course: natural number, whole number, rational number, real number, function, inverse function, series, geometric figure, the equality of geometric figures, vector, equation, inequality, simultaneous equations and inequalities.

As a result, in the various versions of the school course there appeared absolutely correct, strict (from the point of view of "naive" set theory) definitions of natural number as the class of equipotent sets, of whole number as the class of equivalent pairs of natural numbers, of rational number as the class of equivalent pairs of whole numbers, of real number as the fundamental series of rational numbers or a Dedekind section in the set of rational numbers.

The set theory approach to the definition of a function as a particular case of the general concept of correspondence or relation occupied a firm position in school courses. Within the framework of this approach such concepts as reciprocal function and series were given simple formal definitions. Radical changes took place in the teaching of geometry, where the set theory approach not only made it possible to give formal definitions to basic geometric concepts but also led to a significant reorganisation of the entire course, so that to a large extent traditional geometric 'geodetic' problems and theorems gave up their place to the study and use of geometrical transformations. The structuring of geometry in accordance with set theory made it possible formally to define a vector as a class of directed lines (pairs or points) or as parallel translation. Finally, definitional rigorism led to the natural necessity of providing formal definitions for "working" mathematical concepts such as equation, inequality, simultaneous equations and inequalities, and these definitions also appeared in school text books.

The realisation of the notion of definitional formalism in text books did, however, present a number of serious problems for school practice - problems of both a methodological and of a general didactic nature, connected with radical changes in methodics which took place purely as a result of the presence of a strict definition of the concept that was being studied. The fundamental task of methodics with regard to the formation of pupils' concepts consists (in the absence of a strict definition contained in the course itself) in developing in pupils adequate ideas about the concept which are sufficient to enable the pupils to use the concept at an appropriate level. But when there is a formal definition the task of concept formation is to all intents and purposes removed, or more precisely, is transformed into the task of "working the concept through", reinforcing it by means of specially selected examples, counter-examples and systems of exercises.

However, activity on the part of the pupils in the process of reinforcing a strictly defined concept facilitates the real mastery of the concept only in the case where the initial definition is a natural formalisation of the content of the concept, where the definition is, psychologically speaking, matched to corresponding intuitions which are essential for the effective use of the concept in mathematical practice.

Meanwhile the majority of formal definitions of the above-mentioned fundamental concepts tend instead to contradict natural intuitive ideas rather than to agree with them. Perception of natural number as a class of equipotent sets makes sense only on the level of the definition itself and that of the formal structure of arithmetic, but in the applied sense it is completely devoid of purpose. In exactly the same way, perception of whole number as a class of pairs of natural numbers, or of rational numbers as a class of pairs of whole numbers in no way helps pupils to develop sound skills in numerical and algebraic transformations, yet this is undoubtedly one of the most important purposes of early algebra.

The fundamental methodological disadvantage of the set theory definition of function is the psychological discrepancy between the intuitive idea of a function as variable magnitude and its formal definition as a set of pairs, the excessive distancing of the mathematical formalisation of this concept from its "physical" prototype. As for the concept of a vector, perception of it as a class of directed lines accords more closely with intuitive ideas about vectors and the use of vectors in practice than does the definition using parallel translation. At the same time it is interesting to note that methodologically speaking the latter definition is simpler.

Thus the formal definitions of the fundamental concepts, while providing the teacher with some logical and hence psychological comfort, do not provide a solution to the major problem which confronts him: that of inculcating in pupils correct ideas about a concept which are essential for the use of that concept in practical work.

Another no less important methodological aspect of the formulation of strict definitions of the fundamental concepts of the school course is the change in the standard of strictness in the exposition of a theory and its practical applications. Practice in the creation of mathematical theories provides evidence that a theory may develop very strictly even in the absence of strict definitions of its initial concepts (for example the theory of complex numbers up to the 19th century, the theory of numbers before the axiomatisation of the concept of natural number and the construction of corresponding models of whole number) but the reverse is impossible, or at least, highly undesirable.

The level of rigour of the bases of a theory necessarily pre-determines the level of rigour at which it must be studied as a whole; however, when this principle is put into practice in school it inevitably leads to a displacement of the basic emphases of teaching: the study of the concept as such occupies an unjustifiably prominent place by comparison with the use of this concept in a theory and in the solution of problems, and the purely logical aspect begins to dominate the applied aspect to a significant extent.

On the other hand, the differences in the levels of strictness of the definition and the subsequent study of the concrete concept may totally destroy any positive effect from the point of view of the development of mathematical thinking and pupils' general perception of mathematics as an academic subject and even as a science, in so far as for pupils a corresponding extrapolation is completely natural.

It must also be pointed out that an exposition of a theory based on strictly defined initial concepts requires an adequate terminology and symbolization, which are usually very difficult for pupils to master, and the difficulty of which, moreover, is out of all proportion to the objective complexity of the theory which is being studied.

Thus an analysis of the methodological aspects of definitional formalism must take into account the three above-mentioned drawbacks: the psychological disparity between formal definitions and intuitive ideas about mathematical concepts, the unavoidable shift of emphasis in teaching from the applied to the logical, and the need for pupils to master an extensive system of terms and symbols.

These three drawbacks are undoubtedly of an objective nature; they arise out of the very essence of definitional formalism. However in the practical application of the ideas of definitional formalism there were also some cases of subjective methodological exaggerations of the possibilities it provided. This is true, for example, in the case of the formal definition of an equation as "an equality with a variable, for which it is necessary to find all the values of the variable which turn it into a proper numerical equality". Obviously, however, from the logical point of view this combination of words cannot serve as the definition of a mathematical concept in that it brings in not only the objective side of the phenomenon (equality with a variable) but also an obvious subjective element - the purpose for which the equation is being considered.

This disadvantage in the cited definition is in fact quite irremovable, since the subjective element - the point of view of the researcher, his interpretation of the phenomenon - really is the main factor which turns an equality with a variable into an equation, and this element can only be got rid of by dispensing with the term "equation" itself, by identifying it with the term "equality with a variable". This is in effect what has happened in the case of a parallel theory - the theory of inequalities, where there is no special term at all corresponding to the term "equation". To give a formal definition to the fundamental concept of an equation would mean giving it the status of a mathematical concept, a mathematical object, whereas this concept is not in the strict sense mathematical in that its deep structure is of the type "object plus purpose of investigation". The term

"equation" - in the sense in which it is used both in mathematics and in the teaching of mathematics in schools - relates essentially to the metalanguage of mathematics, the language of teaching, the language of descriptions of mathematical activity. The cited analysis of the "equation" shows, in our view, that there are certain limitations to definitional formalism as a means of studying and using the fundamental concepts of the school course.

At the same time definitional formalism, the tendency towards the formulation of strict definitions of fundamental concepts from the school mathematics course, is not so much an independent factor, rather it is a manifestation of a general move within modern teaching methodics in the direction of greater strictness in the organisation and presentation of school mathematics. Therefore reasonable use of strict definitions cannot fail to facilitate the achievement of one of the chief goals of mathematics teaching - the development of logical and generally abstract thought, and this is an undoubted (and well known) merit of definitional formalism. The real effect of this merit, however, can only be achieved through a high level of methodological skill on the part of the teacher, who must be able to overcome the above-mentioned disadvantages of the formal approach to fundamental concepts.

On the other hand, no traditional approaches, however "outdated", to the treatment of fundamental concepts ever present an obstacle in the senior classes - an important stage of schooling, when pupils have already mastered these concepts on the level of content - to the addition of a logical "gloss" and the introduction of strict definitions, for example, in terms of set theory. Such a structuring of the school course will significantly enrich the content of the important section on "the logical structure of mathematics" and moreover - and this is particularly important - it will demonstrate to pupils the solidity of the foundations of the knowledge of mathematical science, foundations which no methodological simplifications used in the presentation of mathematics as an academic subject can ever shake.

In his opening remarks, the author suggested that in the Soviet Union the idea that "the proof of the pudding is in the eating" was lost in the 'Kolmogorov' reforms of the school mathematics curriculum; since then, teachers have been concentrating on diet rather than on food. Definitions based on set theory were inadequate because they did not link up with intuitive ideas. Pupils have spent time on minute study of mathematical notions rather than on their applications. It is evident that this drive towards rigour was misdirected. The important mathematical ideas of number, function, vector, etc, are largely independent of

their rigorous definition; the 19th century formulations about the nature of numbers were not strictly mathematical theories. Further progress in content and methods of school teaching should be reorientated towards developing logical abilities of children in terms of real mathematical problems. The development of logic is not the only aim to pursue - algebra and analysis (as well as geometry) are also interesting because they enable a range of abilities to be developed. We have reached the paradoxical situation where the teachers of mathematics are urging less rigour and the teachers of methods demand more. We need to find a common position.

In discussion, British participants pointed out that 'definition' is a theme which has had little attention recently in the UK. There has been a movement away from the old-style "DEFINITION...", and we now have "definition by stealth", or "definition by osmosis". Reference was made to the work of H J Vollrath in Germany, who has tried to distinguish different levels of ideas which call for different levels of treatment. First there are concepts like function, which one should not attempt to define at school. Second are concepts such as congruence and vector, which merit a slightly more formal treatment, but whose study should not begin with a definition. Then there are concepts such as hexagon and perpendicular bisector, which admit of a formal definition from the start.

This point was taken up in terms of "fundamentals" and "derivatives". Certainly students at school should not be told that a function is a binary relation, or that a series is a function. What is more natural is to build a final stage into the course, "the logical structure of mathematics", where the pupils could have a chance to hear about these logical notions, and where set theory could come into its own.

Reference was made to a quotation from the paper: "Reasonable use of strict definitions cannot but facilitate the achievement of one of the chief goals of mathematics teaching - the development of logical and generally abstract thought." But doubt was cast on this; definitional formalism appeals to students - at an appropriate level of maturity - who have already developed logical and abstract ways of thinking, but it was questionable if the order could be reversed. For many children there is a half-way stage, in which they have to justify statements, but not in a strictly axiomatic, professional sense.

It was pointed out that the process of developing a definition is a continuous one, of which the formal definition is only the final stage. For example, pupils learn that the external angles of a polygon have a sum of 360° . One can then ask: what happens when you have a crossed polygon? This shows the need to say more precisely what we mean by "angle" and by "polygon", and thence to formal definition. Again, in the first instance "rotation" for

children can only take place about an axis within the figure. Later, discussion of the relation between other figures leads to extension of the definition of rotation to apply to the rotation of the whole plane. This is a good kind of mathematical activity, in which the pupils themselves can take part in formulating and refining definitions.

THE BALANCE BETWEEN PURE AND APPLIED MATHEMATICS

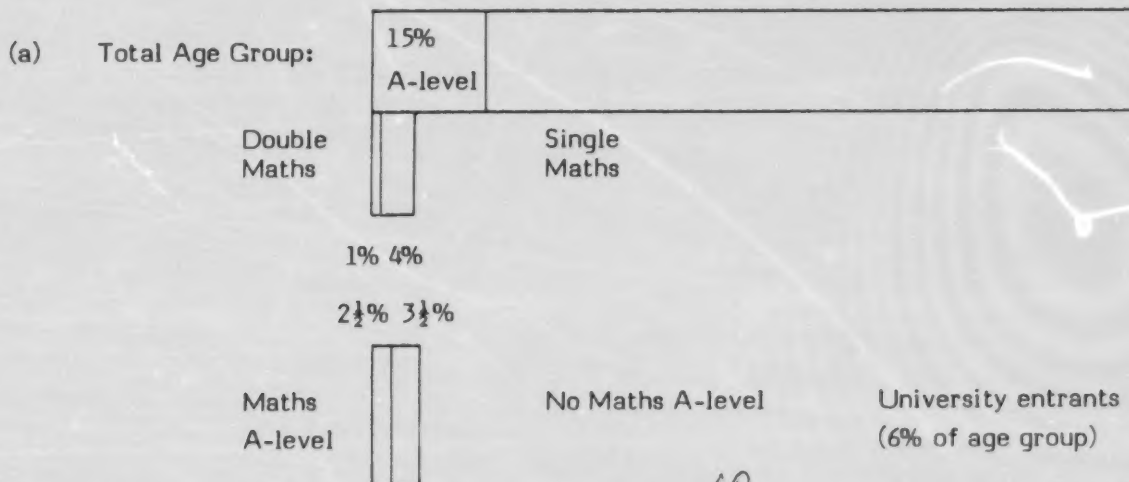
F R Watson, University of Keele

In this paper I attempt to examine the interconnections between theory and application in mathematics teaching, the balance between them and the extent to which the teaching of mathematics is oriented towards application. This topic may be treated on (at least) three levels; I propose to examine each of these in turn. All are relevant to the teaching of mathematics in school.

(1) The teaching of applied mathematics to more able 16-19 year olds.

It has been traditional in Britain for mechanics to form a part of mathematics at Advanced level (the 18+ examination for the General Certificate of Education). In a similar way, University mathematics courses have normally contained a substantial 'applied' component (mechanics of particles and rigid bodies, hydrodynamics, electromagnetic theory and sometimes relativity and quantum theory), which in many (most?) other countries would be regarded as a branch of Physics.

Advanced level examinations are taken by about 15% of the age group (most pupils take 2 or 3 A levels) - of these about $\frac{1}{3}$ (4% of the age group) take (single-subject) A-level Mathematics, and about 1% take 2 A level subjects, Mathematics and Further Mathematics: $\frac{1}{4}$ % go on to read Mathematics at university (with equal or larger proportions going to Physical Sciences, Engineering, Medical/Biological, and 'other' degree subjects respectively). (Fig.1) (All %'s relate to the total age group of 18-19 year olds; sources, the 'Cockcroft' Report (1982), and Moody (1982)). Although, as will be clear, these figures refer to a small proportion of the age group, they do serve to indicate the central position of mechanics within the study of mathematics at this level.



(b) University Entrants:
with any Maths
A-level

28%	17%	12%	21%	22%
Engineering and Technology	Physical Science	Maths (including computing)	Medical, Dental, Biological	Other (including business studies, geography)

Fig 1.

- (a) Total age group - A-level and Mathematics A-Levels.
(b) University entrants with Maths A-Level. (Figures refer to 1979).

A study of A level syllabuses and question papers illustrates a change which has taken place over the past 20 to 30 years - the increasing emphasis on Probability and Statistics at this level, which has intensified the problems of balance and emphasis already present. For now the issue arises whether students should study both mechanics and probability/statistics, or only one - and if one, which one? It is interesting that two recent published opinions on this suggest rather different solutions - the Cockcroft committee (Cockcroft (1982)) felt that:

".....all A-level mathematics courses should contain some substantial element of 'applied mathematics' so that all who are studying the subject, whether for its own sake or because of its usefulness as a 'service subject' for their other studies, are able to gain a balanced view of mathematics. It follows that we do not favour single-subject A-level courses which consist of pure mathematics only". (p171).

The Committee acknowledges the difficulty of trying to do both mechanics and probability and statistics, and does not advocate this approach, whilst feeling unable, "because of the diversity of students' future needs and interests", to recommend the study of either to the exclusion of the other. Schools must make a choice according to their individual circumstances, and where both strands are being successfully incorporated, this should not be discouraged (p173).

In contrast, the Mathematical Association (M.A.) sub-committee discussion paper on the "Applied" content of Mathematics at A-level (Moody (1982)) writes:-

"The basic question is, then, should we teach mechanics, statistics, or both to each student?

We feel that all those who complete an A level mathematics course should have studies something from both areas - that they should experience both deterministic modelling (in mechanics) and non-deterministic (in statistics)." (p6).

(Note that these comments refer to single-subject A level mathematics - where Mathematics and Further Mathematics are studied in a 2-subject A level, the problem of choice is less acute, and here the Cockcroft committee recommend the inclusion of both - p173). There is a current debate about a proposed "common core" of pure maths for single subject A-level which is also relevant to this discussion: SCUE/CNAA (1978); Cockcroft (1982), p175.

The M.A sub-committee discussed and rejected the possibility of options within the syllabus, allowing teachers (and schools) to specialise in mechanics, statistics or a mixture as they prefer. They list as advantages: depth of coverage of one application, student choice of an option related to intended careers, matching the course to teachers' expertise (to which may also be added the possibility of teachers making a gradual transition eg from teaching mechanics to the inclusion of an increasing proportion of probability/statistics as their experience and confidence grows), and as arguments against options: avoiding too early specialisation, career intentions frequently do not commit students to either option, practical difficulties of staffing both choices, and the likely separation of pure and applied sections of the course.

Two final points: first, neither mechanics nor statistics is an easy subject to master in its early stages, and a certain amount of exposure to the ideas of mass, not, acceleration,... as of sampling variability and significance testing, is necessary before pupils are 'at home' with them. This suggests that a 'smattering' of knowledge of either is likely to be of little use.

Secondly, there are other areas of application of mathematics besides mechanics and statistics - in AC-circuit theory, in economics and in geography, which are of growing importance; these have not yet had much impact on the curriculum. "No serious attempt has been made to develop any extended work ..(on such topics)... nor does there seem to be any great demand for such a development" (Mathematical Association, (1982), p5). If such changes do arise, they will intensify the problems discussed above.

(2) The teaching of mathematical modelling

Though it is usually seen as an undergraduate (perhaps even a post-graduate) activity, attempts have recently been made to introduce this topic into the curriculum of schools.

The Schools Council Sixth Form Project at Reading University, directed by C P Ormell, ran from 1969-1976 and culminated in the production of a series of text books, "Mathematics Applicable": Schools Council Sixth Form Mathematics Project (1975-8). The materials were aimed at non-specialist 6th form students (aged 16-18) - biologists, scientists, geographers who were not studying mathematics at A level (an appropriate examination was instituted at OA level, intermediate between O and A levels). Though the project has not achieved wide acceptance it has several features of interest and produced some very interesting material.

One of its themes was "modelling".

"Mathematics is centrally concerned with the investigation of the possibilities of the real world. The relevance of mathematics is to hypothetical, rather than to actual, situations. A naturalistic account of mathematics on these lines treats the symbolism of mathematics as a form of apparatus which we use to construct models of situations. Manipulation of the mathematical model is broadly equivalent to exploring the possibilities of the situation represented by the model." (Ormell (1972)).

A major intention was to show the relevance of mathematics by illustrating it in application - indeed the textbook series is named 'Mathematics Applicable' ('Applied' carries with it the connotation of 'mechanics' - a wider view of application was intended.) Thus a group of situations might be considered which were illustrations of a quadratic model, $y = kx^2$. Later this might be widened to consider $y = k(x-a)^2$, $y = k(x-a)^2 + c$, etc.

The common properties of the situations would be considered, and the mathematical equipment needed to deal with them assembled, motivated by the obvious applicability of the techniques being learned. This curriculum project also introduced a (difficult) problem-solving examination paper (with 'hints' which could be used, incurring a mark penalty), and assessed course work, in the form of projects or mathematical 'essays' modelling some application of mathematics.

Some of the titles submitted were:-

The economics of a bus transport system

Population fluctuations

Two-card pontoon

Is a mission to Jupiter possible?

To rent or buy a TV set?

The best depth of a reservoir
The economics of electric light bulbs.

Course work might take one of four forms - modelling a situation, investigating a proposal, investigating a real object or machine, or looking for specific types of model.

"Why are coursework essays required in the scheme? nearly all the problems in the... textbooks have been softened up before being put in..... It is an important experience for each person doing the course to realise how much thinking is needed to convert any problem from a standing start into mathematical language" (from "Advice on Coursework", an informal Project paper, c.1974).

Some of the ideas were perhaps too difficult for a non-specialist group. Indeed, the materials would form, in my view, a good basis for an A-level course. Administrative and practical difficulties with the examinations also hampered the growth of school involvement.

I believe the Ormell project would repay more study than we have time to give it here. However, I turn now to some more recent developments which aim to introduce mathematical modelling explicitly into the curriculum, as an activity in its own right. Hitherto this has mainly been associated with the specialist study of mathematics at undergraduate level, where a good background knowledge and an appropriate maturity of approach can (usually!) be assumed. However, even at this level, it is difficult, and what experience there is seems to indicate that students can only 'model' effectively using mathematics which they already know well. (Note that this, to some extent, runs counter to the Sixth form Project topics.) Some University tutors have suggested, for example, that there may be a 1-2 year gap between the content levels of topics which undergraduates can deal with (a) in their own right, as pure mathematics and (b) in fluent use in application (ie 3rd year modelling is best done using 1st year mathematics).

Burkhardt (1977, 1981, Burkhardt et al (1981)) has attempted to develop a resource bank of situations suitable for teaching modelling skills in the main secondary age range, say 13-16 years. Examples of such situations are:-

- 1) Design of a car park - maximise parking places?, model cars by rectangles... how to allow access? How many exits, lanes? Accessibility versus maximum number of cars...
- 2) Comparison of the cost of travel to school by bus and by moped (motorcycle).
Number of days in school year, total bus cost (adult fare?). Initial motor-cycle cost, insurance, tax, repairs, resale value... Hire purchase? Non-school use, effect of weather (too wet to go by motor-cycle), danger?...

Further examples are school meals (minimise queues), school trips, running student elections, siting pedestrian crossings, buying school supplies, making curtains, shopping around for a 'best buy'. In Burkhardt (1978, 1981), Burkhardt et al (1981), the structure of the modelling process is analysed as follows (figs. 2, 3 and 4).

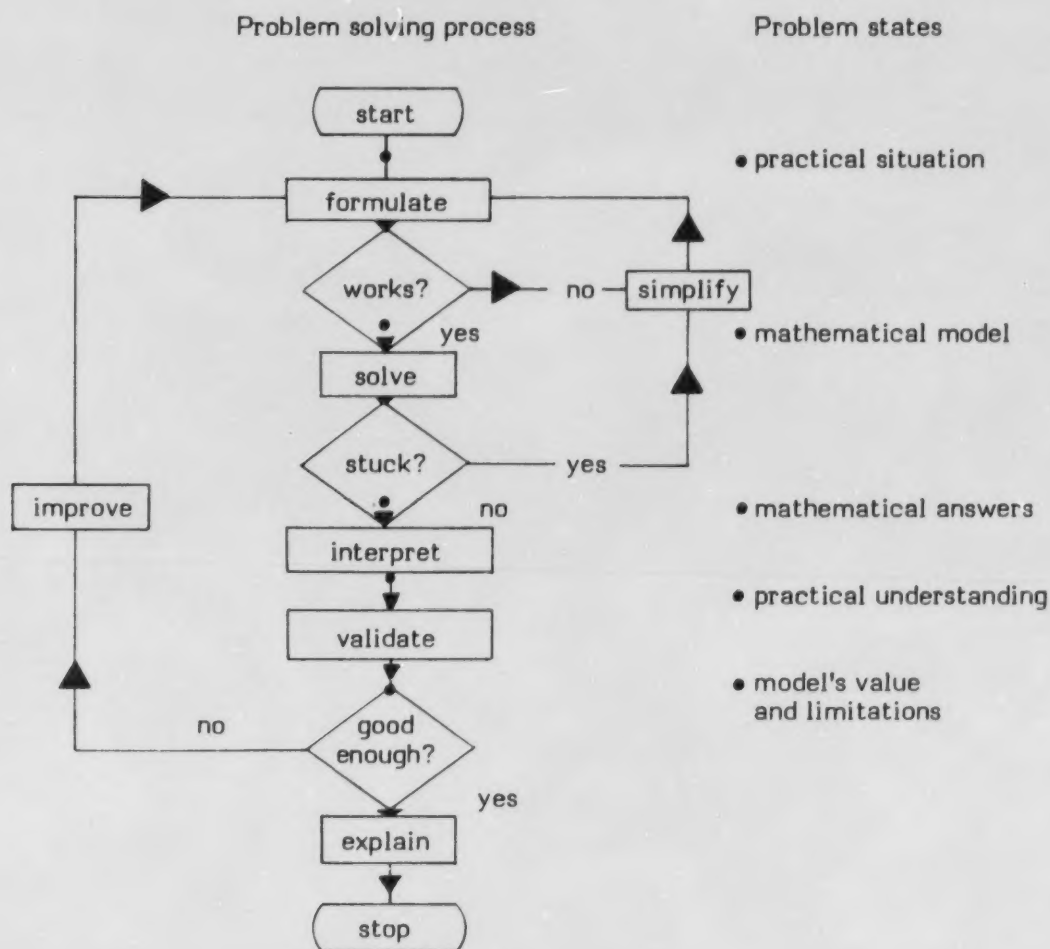


Fig.2 The modelling process: Burkhardt (1981)

Fig. 3 Modelling sub-skills suggested by Burkhardt (1981):

- (1) Generating variables
- (2) Selecting relationships
- (3) Generating relationships
- (4) Identifying specific questions
- (5) Modelling objects and situations
- (6) Estimating
- (7) Validating

Note: (2) and (3) may involve graphical or symbolic representations, and translation of these to and from verbal descriptions or tables.

Figure 4. "Translation skills" in modelling - from Burkhardt (1978)

From \ To	Verbal Descriptions	Tables of Numbers	Graphs	Algebraic expressions
Verbal Descriptions	-	Measuring	Sketching	Descriptive modelling
Tables of Numbers	Reading ↓	-	Plotting ↑	Fitting (Descriptive modelling) ↑
Graphs	Interpretation of Graphs ↑	Reading off	-	Curve fitting ↓
Algebraic Expressions	Formula recognition ↑	Tabulating or Computing	Curve sketching ↓	-

"These skills can be greatly improved by explicit teaching involving practice exercises; it is worth devoting time to this part of the teaching of modelling. Obviously, some skills are more demanding and some of more importance than others. The arrows in the table indicate useful indirect routes - for example the verbal interpretation of algebraic expressions is often facilitated by plotting their graphs, which in turn involves the preparation of tables of numbers." (Burkhardt (1978)).

Thus the ability to use and interpret graphs is seen as a component modelling skill (that of translation) which might be taught explicitly - see Janvier (1978), Swan (1980). My own very limited experience, with teachers in training (using "Beginning to tackle real problems" Burkhardt et al (1981)), indicates that this approach to modelling provokes a lot of discussion and a lot of interest; it certainly brings out the ingenuity and expressiveness of students. Clearly, some mathematics lessons in schools would be very different from their normal pattern if such activities were more widely used; the very nature of mathematics, seen by many teachers and pupils as an exact discipline, admitting only right/wrong answers, would be changed in a situation where assumptions must be made, variables incorporated in the analysis or rejected as of small effect, parameters and data estimated, and the answer tested against reality. The increasing interest in this area over the past few years (see for example, Nelson (1977), d'Inverno and McLone (1977), Burkhardt (1978)), has been evidenced by the recent establishment of a Journal of Mathematical Modelling for Teachers, edited by D N Burghes and originally produced by

the Cranfield Institute of Technology; some of the material is relevant for the 16-18 age group. (It is now published by the Institute of Mathematics and its Applications - the first issue contains an interesting article on our theme by John Hersee (1982). Burghes (1982) has also recently edited collections of similar material for teachers, and, coincidentally with our meeting here, September 1982 also sees an IMA-sponsored conference at the University of Bath on "Teaching Mathematics and its applications" (at University and Polytechnic level).

3. Realism in main school mathematics

The first two sections of this paper have been concerned with what are arguably minor aspects of the school mathematics curriculum - the first one, though relating to an established part of work in schools, involves only a small percentage of the age group, and the second, though an important direction of development which many of us would support strongly, has not yet made any appreciable impact on the situation in schools. This third section considers the curriculum of the average pupil in the main years of secondary schooling (11-16), in the light of our theme - the balance between theory and application. The major source document here, of course, is the "Cockcroft" Report (1982). It will be known that the Committee of Enquiry was set up to examine the teaching of mathematics in relation to the needs of employment and of adult life -and there was concern in some quarters lest the Committee might be under pressure to take a narrowly vocational view of the aims of mathematics teaching in schools.

In the event, the Report solidly endorses a very different approach - the so-called "back-to-basic-skills" movement is firmly rejected and the promotion of enquiry, rather than routine practice of manipulative techniques, is given strong support. Indeed in its list (1982, p71) of six desirable features of a mathematics lesson, the Committee is perhaps somewhat 'in advance' of the general view held by teachers, and many of us would feel it has helped to give public opinion a 'nudge' in the right direction - particularly helpful, since as a broadly-constituted and authoritative committee, it has supported ideas, such as investigation and project work, and discussion between pupil and teacher, and between pupil and pupil, which have sometimes met with scepticism or apathy.

However, the Report does make frequent reference to ways in which the mathematics curriculum ought to be linked with the real world.

In the early years the importance of practical work is underlined:

"... (practical work) is essential throughout the primary years if the curriculum is to be developed in the way we have indicated" (p84).

"... the importance of creating positive attitudes to mathematics did not always seem to have been realised. Even in schools in which the general atmosphere was lively and supportive, the need of children to work at mathematics in practical ways had not always been realised, so that attitudes to mathematics in some classrooms contrasted strongly with attitudes towards other work. In some classrooms children were working at abstract calculations with numbers beyond their experience; their need to measure, weigh and pour water, to count real things and to learn about hundreds, tens and units with apparatus and with money had gone largely unrecognised."(p102).

"... At all stages the teaching of measurement should be based on extensive practical work, including the use of a variety of measuring instruments, the use of compasses and other drawing instruments, and the construction of geometrical shapes and models. All children at all levels need experience of this kind. Some children will acquire facility in measurement and in the use of drawing instruments without difficulty but others will require a long time and much practice to achieve a reasonable competence. This time needs to be provided over a period of years."(p79).

"... In infant classes it is essential to have plenty of small objects which can be used for sorting and counting. There must also be equipment that can be used for measuring, including sand, a sink with a variety of capacity measures and simple balances with appropriate weights. Older children need other measuring devices such as rulers and tape measures, balances and scales of various kinds, and timing devices. Young children need to use simple structural apparatus of various types, such as those in which units can be physically combined into larger pieces or in which collections of units can be represented by larger pieces. At a later stage a different type of structural apparatus should be available which provides a physical representation of units, tens, hundreds and thousands. Apparatus of this kind can assist children to understand the concept of place value. Real or plastic coins should also be available; these can be used both to develop familiarity with handling money and as another form of number apparatus." (p185).

"... All children in primary schools need to use a variety of two- and three-dimensional shapes. There should be a supply of simple drawing instruments including shape templates, rulers, protractors and compasses; children also need to

use plumb-lines and spirit levels. Paper printed with grids of various shapes and sizes is necessary for some kinds of graphical work as well as for work with patterns and shapes. Its use can also help to develop the concept of area, as can work with 'pin-boards' and 'peg-boards'. Some electronic calculators should also be available. All of this equipment, together with such things as scissors, coloured paper, card, elastic bands, string and glue, needs to be easily available when required and careful thought should be given to the way in which it is organised and stored." (p185).

Visits by pupils to companies and other employers, teacher attachments to industry, and visits to classrooms by representatives of industry and commerce are encouraged. (p31). The importance of links with other curriculum areas is also stressed at both primary and secondary levels:-

"... The experiences of young children do not come in separate packages with 'subject labels'; as children explore the world around them, mathematical experiences present themselves alongside others. The teacher needs therefore to seek opportunities for drawing mathematical experience out of a wide range of children's activities. Very many curricular areas give rise to mathematics. Measurement and symmetry arise frequently in art and craft; many patterns have a geometrical basis and designs may need enlarging or reducing. Environmental education makes use of measurement of many kinds and the study of maps introduces ideas of direction, scale and ratio. The patterns of the days of the week, of the calendar and of the recurring annual festivals all have a mathematical basis; for older children historical ideas require understanding of the passage of time, which can be illustrated on a 'time-line' which is analogous to the 'number-line' with which they will already be familiar. A great deal of measurement can arise in the course of simple cookery, including the calculation of cost; this may not always be straightforward if only part of a packet of ingredients has been used. Many athletic activities require measurement of distance and time. At the infant stage many stories and rhymes rely for their appeal on the pleasure of counting." (p95).

"... When considering 'mathematics across the curriculum' it is an over-simplified view to have regard only to the 'service' aspects of mathematics, though these should not be neglected. Those who teach mathematics should be aware of the mathematical techniques which are required for the study of, for example, science, geography, craft or home economics and make provision for them. They should also

try to arrange that the mathematics course and the courses in other subjects are developed in such a way that pupils will be familiar with the necessary mathematical topics by the time they are needed in other curricular areas. Furthermore, there should be liaison between teachers so that those who make use of mathematics in the teaching of their subjects do not use an approach or a language which conflicts with that which is used in mathematics lessons. Conversely, those who teach mathematics should be aware of the ways in which mathematics is applied within other subject areas and should ask their colleagues who teach other subjects to provide examples of the applications of mathematics which can be used in mathematics lessons. Too little attention is often paid to this aspect. Similar considerations apply to applications of mathematics which lie outside the school curriculum. Use should be made in mathematics lessons of material gathered from newspapers and other printed sources as well as of information which pupils are themselves able to provide." (p148).

Again, at the level of the secondary school, it is vital to see mathematics in a context of applications.

"... We believe it should be a fundamental principle that no topic should be included unless it can be developed sufficiently for it to be applied in ways which the pupils can understand. For example, we see no value in teaching and examining, in isolation and as a skill, the addition and multiplication of matrices to pupils whose knowledge of algebra and geometry is not sufficient for them to be able to appreciate contexts within which matrices are of use. On the other hand, some pieces of work may claim a place in syllabuses for higher-attaining pupils because of needs beyond the age of 16". (p133).

"... the ability to carry out a particular numerical operation and the ability to know when to make use of it are not the same; both are needed. The mathematics of employment and of everyday life is always mathematics in context and is based largely on measurements of many kinds made in many different situations. Arithmetical skills are therefore a tool for use in situations which require an understanding of other areas of mathematics, for example the geometry of shape and space and graphical representation of various kinds. An excessive concentration on the purely mechanical skills of arithmetic for their own sake will not assist the development of understanding in these other areas. It follows that the results of a 'back to basics' approach (as we understand the words) are most unlikely to be those which its proponents wish to see, and we can in no way support or recommend an approach of this kind." (p80).

Evidence from an earlier survey by HMI, reported in "Aspects of Education" (DES, 1979) included both praise and criticism of the degree of realism found in mathematical classrooms.

One of the questions raised was: "Is there competence in realistic arithmetic?" "The word 'realistic' is important. Success in repetitive exercises on (say) technical points involving the manipulation of fractions or decimals was, in itself, not taken as sufficient evidence of competence. Competence demanded a certain ability to apply skills, and to choose a skill which was appropriate". (p126).

A report on one school stated:

"... In another lesson pupils had been taught to use square root tables but found the question "Can you work out an approximate value of the square root of 10 without using your tables?" very strange. Applications referred to by teachers were invariably of the textbook exercise type; problems from everyday sources, whose point could be seen by the pupils, were seldom presented. The teaching approach was resulting in the withering away of common sense". (p138).

Three specific questions were examined:

Is the teaching positively related to:

- (i) the calculations required in science?
- (ii) the calculations required in craft?
- (iii) the use of mathematics in other subject areas, eg social science, geography, etc?

In all three cases the answers were 'No' in 70-80% of schools, with 'Yes' and 'Undecided' making up the rest about equally. "There is clear evidence of a pronounced lack of structural interdepartmental discussion about applications of mathematics across the curriculum in most schools" (p149).

"... Not infrequently science teachers attempted unilateral solutions by managing without mathematics as much as possible. This is understandable, and there is a positive side to this approach, but it cannot be the best solution for the pupil in the long term. Coordination was often made difficult by having different setting arrangements for the two subjects; there would not appear to be any simple solution to this problem which would be acceptable to both departments."

There were some exceptions:-

"... A school was visited in which quite exceptional mathematical work was done in the rural studies department in connection with the raising and care of livestock. Records were kept of the sow and the ewes and then of the young animals. Two lambs were born during the inspection, and they were promptly weighed and 'adopted' by the pupils. Accounts were kept of the costs of the various activities, and no single activity was allowed to run at a loss for very long. Mensuration was involved in the planning of enclosures and the provision of feeding troughs. Some of the slower pupils could make no claim to a knowledge of tables yet all could cope with elementary calculations by some method or another. They got there in the end, however slowly, and "I can't do it" was never heard. In the same school some girls were learning, as part of their mathematics, to enlarge skirt and dress patterns from small diagrams, and they were later to make them in the needlework department. There was further liaison with art and weaving." (p150).

"... The contribution which science teaching can make to general mathematical development was exemplified by a particular interesting lesson in which the tension in a wire was increased by the addition of 100g masses, and the pupils' grasp of place value was reinforced by the deliberate substitution of a 1kg mass each time 10 x 100g masses had been added, the whole process being talked through with the teacher." (p151).

"... In a mixed comprehensive school probability and inference were taught in mathematics and put to good use in biology 'genetics', in chemistry, in careers and in religious education (where surveys were conducted) and most notably in English. Here the pupils made block graphs which illustrated the complexity of language to be found in various newspapers. The graphs were then used to infer attributes of their probable readership." (p152).

Commenting on the relationship with other subjects and the world outside, the Inspectors found that:

"... Whatever the mathematics syllabus, the ideas were, far more often than not, presented in a very narrow way. This was so even when the range of topics, as with some modern syllabuses, was large. Topics which had been introduced by textbook writers because of the opportunities they offered for discussion and exciting development were reduced to routine exercises, and the teaching aspired to do

little more than cover the skills required. Illustrations or applications of the syllabus which were not examined were rarely taught. Mathematics was too little related to the world outside and to other subjects in the school, even when these subjects either needed to apply mathematical techniques directly (as in science), or when they gave broader illustrations of ideas which were explicitly on the syllabus. The lack of success in bringing out the purpose of the skills and techniques studied in mathematics is not due intrinsically to the syllabuses in use (traditional or modern); it is far more a matter of the way in which syllabuses are interpreted. It was in non-examination courses, and in some cases CSE courses, that the clearest attempts were made to base teaching on topics from the world outside." (p159-60).

Many mathematics teachers naturally feel insecure at stepping outside what they perceive as their limits of competence - they may have no experience of industry or commerce, and little knowledge of other subject areas such as geography, economics, biology, physics, etc, in which their collaboration is sought. The Mathematics in Education and Industry Committee of the Mathematical Association has, over a period of many years, provided opportunities for school teachers to make contact with industrial enterprises, sometimes by spending a term's leave of absence visiting or working in industry, examining the application of mathematics in use. (The MEI syllabuses at O and A level had their origin in similar activities in the early sixties.) Further developments in this area would be beneficial, but such contacts are costly in money and in time - as are also the smaller scale liaison activities across curriculum boundaries within a school. There seems no easy solution here.

Perhaps I may add two comments in concluding this section.

1) When we speak of realism, it seems important that we should use this in the sense of relating to the world of the pupil. Thus a 13-14 year old may not be interested in 'computations concerning house purchase, payment of taxes, or household bills. These are not yet part of his world, and a project on pocket money or keeping a pet may be more appropriate than production at a factory, or running a car.

Similarly, puzzles and games, though distant from real life in adult terms, may be very much a part of the world of younger pupils - mathematics should relate to the child's everyday concerns so that its relevance and use are apparent rather than taken on trust.

2) There are difficulties in introducing realism- it may demand from the teacher a far wider range of interests and expertise, and may involve his venturing into areas where the pupils have more background knowledge than he does himself. Whilst not a situation to be deplored, this can be somewhat uncomfortable, especially if unfamiliar. Another difficulty is that local or topical material, though more likely to arouse interest, is by its nature ephemeral; material developed in a particular school cannot necessarily be used elsewhere without some labour of adaptation. A discussion of a holiday in Europe, using current prices and currency exchange rates from the newspapers, will (sadly!) be outdated in 12 months time, and fresh material must be gathered. The more vivid and topical the material, the more rapidly it dates, and whilst ideally every lesson should be freshly prepared, it is unrealistic to expect teachers to invest time and energy in producing materials with short life-expectancy.

Some concluding comments

1) There has been considerable discussion in Britain (Bell (1980), DES (A.P.U.) (1981)) of the distinction in mathematical education between product and process, articulated by Bruner "... to instruct someone in these disciplines ... is to teach him to participate in the process that makes possible the establishment of knowledge ... knowing is a process, not a product..."

The desirability of relating mathematics to its applications, (and, indeed, of explicitly teaching the art of applying mathematics - see Lighthill (1971)), suggests the following view of the mathematics curriculum:- (fig 5.)

Figure 5: Mathematics curriculum

	<u>Content aspects</u> (Known methods and results)	<u>Process aspects</u> (More general methods of dealing with familiar and un- familiar situations)
Pure	Rules, algorithms, techniques, eg. solution of equations, differentiation, iterative methods...	eg. generalising, making a conjecture classifying, using a notation... (Displayed in investigations)
Applied	'Laws' of mechanics or probability eg. $P = m.a$, Binomial distribution...	eg. generating variables, selecting relationships, estimating, validating... (Displayed in modelling)

Problem-solving
is defined as
working in an
unfamiliar
situation; it
necessarily
involves both
content and
process aspects

2) The importance of the link between mathematics and its applications has been a theme of many famous mathematicians. "All mathematical development has its psychological roots in more or less practical requirements. But once started under the pressure of necessary applications, it inevitably gains momentum in itself and transcends the confines of immediate utility." (Courant and Robbins (1941)).

I believe it was von Neumann who wrote (though I cannot find the reference):-

"Mathematics derives its inspiration from its applications - if it is cut off from its roots in the real world, it will wither and die..."

Newman (1960) writes:-

"... at a distance from its empirical sources or after much abstract inbreeding, a mathematical subject is in danger of degeneration. Whenever this stage is reached the only remedy seems to be the rejuvenating return to the source - the reinjection of more or less directly empirical ideas." (p2063).

Few today would openly espouse the defiant view of G H Hardy:-

".. very little of mathematics is useful practically, and that little is comparatively dull..."

"I have never done anything useful. No discovery of mine has made, or is likely to make, the least difference to the amenity of the world..."

(Newman (1960), p2029, 2038),

though many mathematicians, and some teachers, do implicitly take up a similar stance by their attitude (and this must be very influential in the classroom).

3) Among the dilemmas which confront us, I refer particularly to one -

In view of the time lag between meeting a new topic, and being able to handle it confidently in applications, should study of application precede the development of the mathematical topics need to handle them, or must it follow?

A compromise solution - both at once - seems untenable; one must come first, although it might be possible to alternate to some extent - ie pose problem, discuss technique, and (later) use techniques in dealing with the problem. (See also Fletcher (1972) for an approach in which new content and applications are admirably blended).

New application areas of mathematics are arising; the growing emphasis in Britain on probabilistic situations has been mentioned above. Simulation techniques seem a likely area of development - a "hobby-horse" of mine - (Watson (1980)); and the increasing availability of the microcomputer will clearly influence the growth of this topic and will have impacts on other applications and their teaching which we can only guess at.

References

- BELL, A.W. Evaluating process aspects of mathematics, University of Nottingham, 1980.
- BURGHES, D.N. (ed) Solving real problems with mathematics, Vols I, II, Cranfield Press, 1981, 1982.
- BURKHARDT, H. (ed) Report of Group M : Mathematical modelling, in University mathematics teaching conference 1977, University of Nottingham 1978.
- BURKHARDT, H. The real world and mathematics, Blackie 1981.
- BURKHARDT, H., TREILIBS, STACEY and SWAN Beginning to tackle real problems, University of Nottingham, 1981.
- COCKCROFT, W.H. (Chairman) "Mathematics Counts ; Report of the Committee of Enquiry into the teaching of mathematics in schools, HMSO, 1982.
- COURANT, R. and ROBBINS, H. What is Mathematics? Oxford University Press, 1941.
- DEPARTMENT OF EDUCATION AND SCIENCE (HMI) Aspects of Education in England, HMSO, 1979.
- DEPARTMENT OF EDUCATION AND SCIENCE (APU), The secondary mathematics survey (1981), HMSO, 1981.
- d'INVERNO, R.A. and McLONE, R.R. A modelling approach to traditional applied mathematics Math. Gazette 61, 92, 1977.
- FLETCHER, T.J. Linear algebra through its applications, Van Nostrand, 1972.
- HERSEE, J.W. Mathematics - Applied, applications, applying Teaching Mathematics and its Applications 1, 1, 21, 1982.
- JANVIER, C. The interpretation of complex Cartesian graphs (Ph D thesis 1978), University of Nottingham, 1978.
- LIGHTHILL, M.J. The art of teaching the art of applying mathematics Math. Gazette 55, 249, 1971.
- MOODY, P.V. (ed) Mathematics at A-Level; a discussion paper on the 'Applied' content, Mathematical Association, 1982.
- NELSON, R.D. Mathematical modelling in the classroom, Math. Gazette 61, 82, 1977.
- NEWMAN, J.R. The World of Mathematics, Vol IV Allen and Unwin, 1960.
- ORMELL, C.P. Mathematics, science of possibility Int. J. Math. Educ. Sci. Technol. 3, 329, 1972.
- POLLACK, H.O. Applications of mathematics (Chapter 8 of Mathematics Education, 69th Yearbook of NSSE) Chicago University Press, 1970.
- SCHOOLS COUNCIL SIXTH FORM MATHEMATICS PROJECT: 'MATHEMATICS APPLICABLE' 'Mathematics changes gear', 'Teaching mathematics applicable' and other texts. Heinemann, 1975-8.

STANDING CONFERENCE ON UNIVERSITY ENTRANCE/COUNCIL FOR NATIONAL ACADEMIC AWARDS A minimal 'core' syllabus for A-level mathematics, SCUE/CNAA, 1978.

SWAN, M. The language of graphs, University of Nottingham, 1980.

WATSON, F.R. A simple introduction to simulation, Institute of Education, University of Keele, 1980.

Mr Watson's paper was illustrated by extracts from 'A' level mathematics syllabuses and question-papers, and also by material from the Ormell and Burkhardt approaches to which he had referred.

In the subsequent discussion, divergent views were expressed about the value of spending time in school on modelling activities. It was asserted that, whilst it is important to teach mathematics with a view to its application, one should concentrate on providing a basis of applicable mathematics. This should first be used to deal with low level day-to-day problems, and then lead on to practical questions of a geometrical nature. Basic mathematical ideas were derived through reflection on practical needs, and through generalisation of processes rooted in reality; but the scope for teaching practical applications in school is very limited. Another suggestion was that applications fitted more appropriately in optional courses, where topics such as optimisation and linear programming could be covered.

The concept of the stability of curriculum content was introduced. There is some mathematical material which can be taught with reasonable success by teachers whose mathematical training is not very strong; this was described as "stable". It was suggested, however, that applied mathematics is an example of unstable material - it should either be taught brilliantly or not at all. By adopting a cautious approach to applications in school there is a good chance that we shall achieve what we are aiming at.

Another speaker made two further points. First, in England many of the new ideas in this area were introduced by individual curriculum developers rather than administrators, and their more radical approach was thought to be beneficial. Second, it was not enough for the mathematics taught to be applicable - it must be seen to be applied.

In summing up, Mr Watson emphasised that many of the ideas in his paper were not extensively implemented in England. Although the teaching of applied mathematics was a well-established practice, very few teachers used the material described in his second section, whilst the third section described a problem that has hardly begun to be solved.

Session 5. Developmental Work in Mathematics Education

MATHEMATICS TEACHING PROJECTS IN GREAT BRITAIN

Dr Alan J Bishop, Department of Education, University of Cambridge

The period 1960-80 was a remarkable era for mathematics teaching in Britain. It was a time of great change, of experimentation with all aspects of mathematics teaching, of developments in cooperation between different 'components' of the system and of growth in the stature of the mathematics teachers' profession. The idea of the 'curriculum development project' was born during that era and projects were both the results of, and the instrument of, change. In the late 1950's there was a great deal of concern about the state of mathematics teaching in Britain, fired not only by internal considerations but also by what was happening in other countries. In America the UICSM and SMSG projects had been gathering momentum, and the OEEC Seminar in Royaumont, France, stimulated much interest in all the European countries. Curricula, syllabuses, examinations, textbooks and teaching methods were all being questioned and pressures for change were coming from all quarters. It was as responses to those pressures that the British projects began. The range of British projects in mathematics teaching is enormous. In 1968 the Mathematical Association produced a "Guide to Mathematics in British Secondary Schools" and in 1976 it was necessary to produce a "Revised Guide". In the Introduction to this later publication it says:

The purpose of this Second Edition is to supercede, not amend, the First Edition of 1968, now out of print. Since then, several older projects have dropped out of the list, their work now accomplished, leaving their mark in the form of established textbooks;..... There are scores of other regional projects, both in mathematics and in computing, which cater for specific local needs and are sustained by the efforts of working groups of teachers Some newer projects are still in the early stage, being cooperative, self-help ventures among a group of schools who are understandably reluctant to endure the added burden of enquiry which is to be expected as a result of this publicity; indeed, they may never in that sense 'go public'.....

Why so many projects?

The key to understanding the projects lies with the highly decentralised British educational system. In such a system, change occurs as a result of individual and group

initiatives rather than being prescribed by a central decision-making body. Our system, which allows school departments to choose which textbooks they will use, which syllabus to allow their pupils to follow and which examinations to enter their pupils for, allowed groups of mathematics teachers to design their own syllabuses and curricula, to request that examinations be set on them, and to develop their own textual materials. Conferences and meetings between teachers, mathematicians in universities and in industry, and teacher trainers did much to encourage the development of ideas. The main professional subject bodies (which are independent of the Government and of the teachers' unions) are the Mathematical Association and the Association of Teachers of Mathematics and they too played their part, though they themselves did not initiate any specific projects. One answer to the question "Why have so many projects?" is therefore that they were a natural development in a period of change, within our decentralised system.

The actual idea of the 'project' seems to have come from industry and from the model for curriculum development used in the USA. The essential features seemed to be that of a team or a group of people working together, perhaps informally and occasionally, towards a common goal, which in our terms would usually consist of the derivation and production of syllabuses, curricula, examination papers, textual materials and teachers' guides (though not all projects produced all of these). Another feature was that most projects' group members were all 'part-timers', in the sense that they would be paid for doing another job, such as teacher or college lecturer, teacher educator or inspector, and their involvement in the project would be on a spare-time, unpaid basis. As projects grew they needed more full-time staffing, and when some later projects started they engaged some full-time staff from their beginning. But the 'flavour' of a British curriculum development project is of a group of hardworking and committed individuals spending much of their own free time meeting and discussing ideas, writing out materials, and then trying them out in their places of work.

In most cases the local nature of most of our projects is caused not just by our decentralised system but also by the need for people to be able to meet together easily, and cheaply. Another answer to the question "why have so many projects?" therefore is a geographical one - in order that the 'group' nature of the project could function easily. For example, the School Mathematics Project (SMP), whose material is now very widely used, began with a small group of teachers in the South of England. Likewise the names of the Midlands Mathematics Experiment, the Manchester Mathematics Group, and the Shropshire Mathematics Experiment all show their local and regional roots.

A third answer and perhaps the most important is provided by the fact that once it is appreciated that no one project could conceivably change our entire mathematics teaching system, then it is necessary to "attack" the system at different points and with different

emphases. Focussing on a specific age group is one way to approach the problem, so we have some projects which aim at Primary children (eg. Nuffield), some at Secondary (eg. SMP) and some only for Upper Schools (eg. Schools Council's Sixth Form Project).

Other projects concentrated on a particular group of children, for example, the 'Mathematics for the Majority' project was aimed at pupils who were unlikely to go on to study mathematics after the age of sixteen. The 'Continuing Mathematics Project' on the other hand was concerned with those pupils, older than sixteen, who were not mathematics specialists but who needed particular mathematical knowledge for application in subjects like science.

Some projects focussed more on changing the methods of teaching rather than just changing the content of mathematics courses. For example, the SMILE project (Secondary Mathematics Individualised Learning Experiment) in London, and the Kent Mathematics Project, contain many individual and group activities, and aim to reduce considerably the teachers' reliance on whole-class teaching.

Other projects concentrated on a very specific area of the syllabus. For example the Project on Statistical Education (POSE) as its title suggests was focussed on developing the teaching of statistics, and the Mathematics in Education and Industry Project, and the Computer Education in Schools Project, indicate their specific concerns.

Other important aspects of the projects

1) Their different financial arrangements. Although it is possible for individuals to give their own time to project work, there are financial limits beyond which their institutions cannot go. The production of pupil materials is a costly business so even if a project begins life in a very small way it is likely that it will need some additional funds to proceed further. In some cases the Local Education Authorities have provided money and facilities: for example SMILE, the Kent and Shropshire projects are all supported this way. Other projects like Nuffield, the School Mathematics Project and the Computer Education in Schools Project received support from industry and business sources.

A national and state-supported organisation called the Schools Council was set up in the early 1960's and this provided the financial support for projects like the Mathematics for the Majority Project, and the Sixth Form Project.

However another source of finance for projects is offered by the sales of their own products. Either in liaison with the established publishing houses or by developing their own publishing facilities, many projects offer, and market, their books and teaching

materials to schools, and indeed to countries, a long way from the original area where the project began.

2) Projects have a life of their own. They are sustained by a perceived need, a commitment by the project members and sufficient funds. So when the perceived need has been met, or the enthusiasm of the members dies, or the money runs out, the project may cease. This does not mean that the project was a bad one or not popular, indeed its material products may well thrive after its own death, so to speak. Among the projects which no longer exist as projects, but which have left still useful materials, we might mention the Mathematics for the Majority and its Continuation Project, the Nuffield Primary Project, and the Midlands Mathematics Experiment.

3) Often as a project grows in scale and influence, and as its materials are used by teachers other than those who were part of the initial project, there develops a need to ensure that the teachers know how to use the materials. Increasingly therefore, projects run courses for teachers, and thereby will 'infiltrate' the formal in-service educational provision. Indeed a project's influence may go well beyond its initial goals - for example, one of the conditions for an LEA wishing to participate in the Nuffield Primary Project was that it should provide a Teachers' centre where the teachers could meet on a regular basis, attend discussions and courses, and prepare materials. Most of those centres still exist and fulfil an important in-service function many years after the original Nuffield project ceased in 1972. They are now an integral part of many LEA's in-service provision.

The 'problem' of effectiveness

The title for this final section is deliberately chosen! Nothing is more likely to cause intense debate in British educational circles than the issue of 'effectiveness' of any aspect of education. For us in mathematics education it is no different. Even if my colleagues could agree with my analysis and description of the range of projects in the first part of my paper they are certain to disagree with my analysis of 'effectiveness'. Any discussion of effectiveness, evaluation, assessment or examination brings out our assumptions regarding the goals and aims of education, and in a decentralised and pluralistic system such as ours there exist widely differing views about those assumptions.

Even to raise the issue of effectiveness in this context can cause annoyance for some people. They would argue that it is the creative and cooperative aspects of curriculum projects which are their crucial features; evaluation is not only inappropriate, but it also spoils the fun!

In actual fact, I believe that this feeling was widespread in the very early projects. Few of them made any statements about specific aims, and effectiveness was therefore judged very personally and very pragmatically, eg. is it interesting to be part of this 'change' process? How do my pupil's respond to the new materials?

Many projects even today satisfy those criteria of effectiveness. The motivation for a teacher to be involved in trying to change part of the educational system is in many cases entirely contained in that personal 'involvement'. The process of changing is evaluated therefore as much as are the products of change, so that when we discuss the effectiveness of projects we always recognise the effectiveness of the process as well as the effectiveness of the products, like the books, the syllabus, the guides and the other materials.

Various criteria of effectiveness have been used:-

- a) popularity and extent of use of the project's materials
- b) public examination results
- c) test and questionnaire results
- d) user/consumer feedback
- e) number of teachers engaged in the project
- f) quality of the ideas embodied in the materials.

As was implied earlier the choice of criteria depends on who is judging the effectiveness and for what purpose. A teacher thinking of joining a project might use (e) and (f) whereas a head of department deciding on which projects' books to buy might look to (a), (b) and (f) for guidance. A project worker might be particularly interested in (b), (c) and (d) and a funding agency might look generally at (c), (e) and (f).

Generally, after the first pragmatic years of the early 1960's, Curriculum Development and Evaluation became an important 'Education Concern' with lectures and courses given on it, articles and books being written, conferences organised and even completely new journals begun (eg. The Journal of Curriculum Studies). Projects began to be understood as the vehicles for curriculum research, and a whole research methodology developed rapidly. It is therefore now impossible to answer, simply, a question like "which projects have been the most effective?"

Projects are now an established part of our educational system and are there to be debated, argued over, compared, chosen between, exalted, condemned, supported or

opposed, in just the same way as we treat any other aspect of our system, whether it is a textbook, a syllabus, a method of teaching, a type of school organisation, a kind of examination question or a topic on the syllabus.

However, in order to avoid accusations that I am running away from my duty, I shall finish this paper by suggesting what I think has been learnt about what makes for a successful project:

- (a) It is important to be clear what aspect of the curriculum the project is focussed on.
- (b) A team is essential and also it should contain more than just mathematics teachers.
- (c) To develop teachers' guides is not enough, pupil materials need also to be developed.
- (d) The examinations need to be considered, and changed if necessary.
- (e) The older the pupils the more difficult it is to change methods.
- (f) Changing teachers' perceptions is as important a goal as producing materials.
- (g) A project should be viewed as an experiment - not as the answer to everything.

References to the projects mentioned in the paper

UICSM - University of Illinois Committee on School Mathematics (USA) 1952.

SMSG - School Mathematics Study Group (USA) 1958

School Mathematics Project - 1961

Midlands Mathematics Experiment - 1962

Manchester Mathematics Group - 1962

Shropshire Mathematics Experiment - 1964

Nuffield Mathematics Teaching Project - 1964

Sixth Form Mathematics Project - 1969

Mathematics for the Majority - 1967

Continuing Mathematics Project - 1971

Secondary Mathematics Individualised Learning Experiment - 1973

Kent Mathematics Project - 1970

Project on Statistical Education - 1978

Mathematics in Education and Industry - 1963

Computer Education in Schools Project - 1967

For other information see:

"A revised guide to Mathematics projects in British secondary schools" Mathematical Association, 1976

"Developments in Mathematics Teaching" F R Watson, Open Books, London.

In introducing his paper, Dr Bishop pointed out that it attempts to answer 3 questions:

Why are there so many projects?

What characterizes a project?

How are projects evaluated?

He stressed that being involved in the process of change is seen as important as the products of change.

Much of the discussion was by way of clarification to the Soviet participants of the role that projects play in the English education system. Among points covered were:

1. Teachers in England have considerable responsibility, and try to select the best for their pupils. Many schools use the materials of one project for part of the curriculum (for example a particular age-range or attainment-group), and the materials of a different project for another. Often in one school several parallel courses could be in operation.

2. It would be a mistake to think that all schools used material developed by projects. For example, it is estimated that about 70% of Sixth Formers (ages 17-18) use texts written by individual authors, rather than produced by projects. Projects often produce ideas which are absorbed by such individual authors; they may also help to train authors, a number of whom gained their early experience through being part of a project writing-team.

3. Parents, and to some extent pupils, have some influence on the choice of teaching materials used in a school; in a good school much consultation with parents may take place, though the ultimate decisions rest with the teachers. The policy of a school with respect to other subjects may also influence the choice of a mathematics project.

4. In response to the remark that the materials of many English projects seemed to contain mathematics that was fragmented and jumped about from one topic to another, Dr Bishop pointed out that we have mathematics courses, rather than "algebra" or "geometry" courses, and usually adopt a multi-topic, spiral approach. Although it may seem unsystematic, it is simply that it is a different system.

5. We do not have agreed, systematic criteria by which to evaluate projects. The whole system is so complex that no project attempts to "solve" a problem in more than one sector. Curriculum development experiments, and attempts to evaluate them, are an on-going process of great complexity.

TEACHING AND METHODOLOGICAL MATERIALS FOR MATHEMATICS

K I Neshkov, Institute for Scientific Research into Content and Methods of Instruction

The resources for teaching mathematics in secondary school are every year becoming more numerous and varied. However one must neither expect nor hope for the emergence of some universal resource which can replace and supplant the rest. This is confirmed by all previous methodological experience and in particular that experience relating to the downfall of unjustified hopes which had been placed, for example, on programmed text books or teaching machines. More serious and deeper foundations underlie the extraordinary complexity of the process of mathematics teaching, which deals with its own unique material and in which people - teachers and pupils, with their endlessly varied characters and not always controllable temperaments - take part. The impossibility of creating a universal teaching resource is also a consequence of the complexity of the problems which mathematics teaching must face and which relate not only to the transference of a particular body of knowledge but also to the development of a dialectical-material view of the world and the proper attitude towards the world which that view determines.

Any new teaching resource appears as a possible solution to a problem that existing resources cannot solve or cannot handle sufficiently effectively. Even as the new resource is evolving it already has corresponding functions which subsequently will not remain unchanged. A teaching resource constructed for the solution of a particular educational problem and a teaching resource which is widely used in practice are not one and the same. A teaching resource is strongly influenced by the other resources in conjunction with which it is used, and also by the methodics of its use. An understanding of the changes in the functional properties of certain teaching resources under the influence of others has given rise to the need to investigate the question of how they function in conjunction with each other, which is particularly important given universal secondary education, and the possibility of which is the result of a significant improvement in the material provision made for our schools. The Institute for Research into the Content and Methods of Instruction and the Institute for Research into School Equipment and Technical Methods of Instruction are devoting considerable attention to the development of series of teaching resources. Thus for example in the Institute for Research into School Equipment and Technical Methods of Instruction, investigation of this question has led to the drawing up of lists of standard equipment and materials for the school mathematics classroom, which can be used, in accordance with the age of the pupils, the topics being studied, the methodological outlook of the teacher and the pupils' requirements, to make up particular

subsets of materials. Also considerable attention is given to questions of how to create the best possible conditions for the organisation of pupils' educational activity in the learning process. Thus each element in the educational series is directed primarily towards the needs of the pupil and is called upon to organise and assist his work in the learning process.

The use of a fairly complex series of teaching resources within universal secondary education, given raised requirements of the outcome of education, is not a simple matter and presupposes mastery of specialised knowledge. And changes in the series itself, in the links between its component parts and in the functions of the individual component parts necessitate constant improvement of this knowledge. Consequently it is essential to develop a methodological series for the benefit of the teacher, which fully corresponds in aims, structure and content with the teaching series. Efforts have been made in this direction in the Institute for Research into the Content and Methods of Instruction of the USSR Academy of Pedagogical Sciences, with some measure of success. Methodological series have been developed for mathematics, to assist the work of the teacher either in individual classes or in groups of classes.

Each of these series includes a syllabus, a textbook which is organically linked to a book of mathematical problems, a methodological handbook for the teacher, didactic material for the pupil, material for work out of class, the theoretical foundations of the course, etc.

A thorough analysis of the teaching methodological series which have been developed, together with their widespread experimental use, have revealed both positive and negative sides. The weakest point is in the co-ordination of elements in the teaching and methodological series (which were compiled by different teams of workers at different times) and also the functional overloading of individual elements in the series. More precise definition of the functions of each element of both series and a tightening up of the links between the series themselves have made it necessary to unite them within different frameworks of a single teaching and methodological series which can fully serve the mathematics teaching process while taking into account the basic needs of pupil and teacher. The term "teaching and methodological" underlines the basic purposes of the series - to ensure not just a learning process or a teaching process but a process of education. It also underlines that which is required of each element in the series, namely that it reflects as fully as possible the teaching and methodological essence with which it is imbued.

The more complex, many-sided and multipurpose the process of mathematics teaching in secondary schools becomes, the more complex the teaching and methodological series becomes, and we believe that it also becomes more essential to pick out those elements of.

the series which are fundamental and determine the educational process. The basic element in the series is fundamentally different from any universal resource in that it presupposes the constant use of other resources which are parallel and supplementary to it, whereas the universal resource by its very nature rejects all other teaching resources.

The basic elements in the teaching and methodological series are the syllabus, the textbook and the methodological handbook for teachers. These essentially represent not only the two sides of the educational process - the pupil and the teacher, but also society itself, which directs this educational process, placing certain goals before it and providing the conditions for the attainment of these goals. Basically this is achieved firstly, with the help of the syllabus, in which the general aims of mathematics teaching in secondary school and the general level of knowledge which all pupils must acquire are formulated; secondly, with the help of the textbook which reveals the content and structure of the material which the pupils must master; and thirdly, with the help of the methodological handbook, which fully corresponds to the syllabus and the textbook and provides the teacher with an exposition of the basic methodological conceptions relating to the use of the textbook and other elements of the teaching and methodological series.

The mathematics textbooks which are being used in secondary schools at the moment are intended - to a greater extent than were those books which they replaced - not as basic but rather as universal teaching resources. This impression is reinforced in that in these, the exposition of theoretical material and exercises are organically linked which was not the case in the previous textbooks. In each such textbook the necessary theoretical knowledge is presented in the form which is convenient for both teaching and learning, and there are exercises of various types which facilitate the formation of mathematical concepts, learning by problem solving, the development of important skills and abilities, the mastery of mathematical language, the development of logical thought and so on. Moreover the textbook determines the logic and sequence with which the course develops, and the scope and level of the material presented to pupils, and it solves a number of problems relating to general up-bringing. But the infinite expansion of the functions of the textbook, the attempt to turn it into a universal learning resource is at the very least unjustified, and given teaching and methodological series of materials it is not necessary. Moreover that would lead to the loss of the textbook's "character"; it ceases to be a textbook for the pupil and becomes a textbook for everyone. Many of the functions which for one reason or another the textbook is supposed to fulfil may be transferred to other elements of the teaching and methodological series so that the textbook retains only those which will be of relevance in classroom teaching conditions. Consequently - and this is not the case at present - the essential and only addressee of the textbook should be the pupil.

An analysis of the use the pupil makes of the textbook at various stages of the learning process helps to clarify its nature as one of the basic teaching resources which constitute the teaching and methodological series. As a result of analysis, the textbook emerges as a short teaching book devoid of any methodological framework which in the best possible way enables pupils to work independently having listened to relevant explanations by the teacher and having carried out (under the teacher's guidance and using different resources) work on the formation of concepts and methods of problem solving.

A textbook of this sort must contain the basic theoretical knowledge (definitions, axioms, theorems and their proofs, formulae, rules, solution methods) which pupils are expected to retain in their memory over a long period. Apart from theoretical knowledge the textbook must include examples of solutions of basic types of problems which the pupil must be able to handle, and also a minimal set of these basic problems. The organisation of the material in the textbook must be such as to facilitate mastery of it. It is desirable to accompany each portion of theoretical material with test questions. With the help of these questions the pupil can constantly check his level of knowledge and the teacher can find out how much of the material the pupil has mastered. The most important of the test questions, if not all, can be used in compiling examinations.

Given universal compulsory secondary education, when large numbers not only of pupils but also of teachers are being involved and there is considerable movement from one class to another, some unification of the methodological side of the teaching process is expedient. In this situation a significant stratum of teachers require more concrete and purposeful methodological help, and the need for this will increase when all the methodological apparatus is excluded from the existing textbooks. Not every teacher will easily and successfully select the expedient problem or the best way to introduce this or that concept, or will be able to draw up a proper system of exercises which satisfy all contemporary requirements, or find and select the necessary historical information and so on. Besides, for each piece of preparation of this sort even a capable teacher would require so much time that he would hardly be able to do it. He must receive help from one of the already mentioned basic elements of the teaching and methodological series. In the same way that the textbook provides work for the pupil to enable him to master the material after corresponding work in the classroom, so the methodological handbook for the teacher must provide the teacher with full and concrete recommendations as to how to run lessons.

The methodological handbook for the teacher must correspond fully to the textbook and in some sense the two together must constitute a single whole. It must contain all the material which is presented in the corresponding textbook, and in the same order. Also it

must have problems, preparatory and test exercises, a system of exercises designed for the purpose of concept formation, and a system of exercises necessary for developing skills and abilities, mastering mathematical language and developing logical thought. This handbook must provide the teacher with information about possible difficulties and the ways of overcoming them, and about typical mistakes that pupils make and ways of averting them. It may be said that the teacher's handbook should enable the teacher to gain a concrete conception of the course of the process that will take place in every lesson even before the lesson has been given.

Mr Neshkov was unable to be present at the Seminar and his paper was introduced by his colleague Mrs Mindyek.

Mrs Mindyek described the development of textbooks and teachers' manuals over the past fifteen years, explaining the difference between the curriculum as planned and the curriculum as actually implemented. The reforms of the 1970's had brought about changes, not only of mathematical content, but also in methodics and in the nature of textbooks. Prior to those reforms there had been two textbooks, one for theory and one for exercises. The new textbooks had combined these functions, with rather less theory than before.

The teachers' manuals have changed greatly, too. They now give more help to young teachers, and also to older, more conservative teachers. The standard pattern is: introductory tasks followed by explanatory material, with each piece of theory being followed by exercises to check the pupils' level of understanding. Earlier teachers' manuals mainly described the 'philosophy' of the programme; the present ones appear to be more like the teachers' guides we know in England.

There was discussion on the purposes of textbooks - are they for pupils' use only, and, if so, should the pupil be able to use them on his own as a self-teaching aid? In Britain we have tried to use textbooks, in an innovatory period, for too many functions simultaneously. The tendency now is to use a variety of books, rather than trying to produce one text to do several different jobs.

THE SCHOOL MATHEMATICS PROJECT (SMP) - A REVIEW

J Hersee, Executive Director, SMP

Editor's note. This is essentially an information paper, written to amplify Dr Bishop's paper on Projects by a more detailed description of one particular project.

Introduction

With no national curriculum body to impose changes in the mathematics curriculum, changes are brought about by:

- new textbooks and teaching materials
- new examination syllabuses for the external examinations taken at ages 16 and 18.

However, the scope of changes that can be brought about by these means is limited and the three aspects of change:

- content
- teaching approach
- examinations

(which are, of course, closely linked with each other) are not equally affected by these means.

Projects have sought to change the first and second aspects, recognising that although examinations can stifle or distort the aims of the reforms, they can also be used as an instrument to encourage change.

The SMP: aims and evolution

The SMP's work began in 1961 in selective secondary schools. The first aim was to produce radically new O-level and A-level⁺ courses. (It is interesting to note that the initial aim was expressed in terms of examination syllabuses.) It was realised that this would involve:

+These are examinations taken at age 16 and at age 18 respectively. The full titles are **General Certificate of Education, Ordinary Level** and **General Certificate of Education, Advanced Level**. Further details are contained in the background paper Examinations and the Curriculum.

- the production of textbooks
- new examination syllabuses
- support for teachers through advice and courses.

In the early years statements about the work of the SMP were usually concerned with changes in mathematical content, but wider aims were also present from the beginning and gradually became explicit. The most obvious features of the first SMP text-books were the new mathematical topics introduced and some traditional topics which were omitted, but the style of the books reflected the author's views about teaching approaches.

Thus, some four or five years after its foundation, the SMP's aims might be summarised as follows:

- the mathematical content of the course should reflect the present-day nature and uses of mathematics;
- there must be unity in the underlying mathematical ideas of the course;
- teaching styles should embody the best of current practice;
- the pupil should take an active part in his learning;
- mathematics should be seen to be useful and presented and studied in context;
- every course should be complete and satisfying in itself besides laying foundations for further study and equipping the pupil with important skills and knowledge.

These broad aims hold today. The different series of SMP texts and other materials have been produced to meet the needs of different groups of pupils and teachers and in response to changing patterns of schooling. When books are revised account is taken of changing needs and conditions, and improvements suggested by the experience of teaching with the original texts are incorporated.

Two other themes to be found in early statements about the SMP deserve comment.

(a) The idea of 'continuous curriculum change' was put forward. The SMP grew out of a reaction against the stagnation of school courses in which there had been little change for many years. It would have been wrong to create a new curriculum, which in its turn, would ossify. Further, it was recognised that the SMP approach to curriculum development was only one of a number of possibilities which should be explored.

In the event, for the SMP, continuous curriculum change has had to be tempered by practical considerations:

- While an element of change is stimulating for teachers and pupils, they, and others, need some stability to give a sense of security.

- It is not so clearly obvious, as it was in 1960, what changes are needed.
- In a small group of schools, with well-qualified teachers, consensus can be achieved; experiments can be tried and rejected if they fail. As the SMP grew and drew in a much larger number of schools there was inevitably much less flexibility.
- The original SMP authors were a talented team with considerable experience of teaching and writing. It should not be assumed that teams of similar quality can easily be found.

(b) It was suggested that the SMP have a limited life - perhaps 10 years. The changes embodied in its courses would by then have been assimilated in others and there would be no need for its continuing existence.

In fact, the SMP continues its work for a number of reasons:

- The changes have not been so widely accepted as had been assumed and, where they have been accepted, there has been dilution and distortion. Changes in **mathematical content** have been incorporated in examination syllabuses and school courses, but examination questions and teaching approaches have not reflected desirable changes in style.
- Experience in teaching SMP courses has indicated desirable improvements in the text-books and teaching materials. The original books were full of exciting ideas, but many of these have proved too difficult for most teachers and pupils. Revised versions tend to adopt a more straightforward approach, making the mathematical ideas more accessible, but with some loss of the original spirit.
- New needs have appeared.
- Further developments have become necessary.

SMP series of textbooks and teaching materials

The main series are:

(i) SMP Books 1-5 The first SMP texts (books T and T4) were written for use in selective secondary schools with pupils aged 13-16. They were followed quickly by, and have been absorbed into, Books 1-5, which provide a course for the most able secondary school pupils, leading to the SMP O-level examination.

(ii) SMP New Books 3, 4, 5 These are a substantial revision of Books 3, 4, 5. Besides incorporating changes suggested by the experience of teaching with the original books, the new books include many more exercises for the pupil and accept the electronic calculator as the standard calculating aid. Suggestions for the use of microcomputers are also included.

(iii) SMP Revised Advanced Mathematics Books 1, 2, 3 Based on the original four Advanced Mathematics books, these books present the course leading to the SMP A-level examination for students who are studying 'single mathematics'.

(iv) SMP Further Mathematics Books 1-5 For pupils who are studying 'double mathematics' these cover the additional work required for the Further Mathematics A-level examination.

(v) SMP Books A-H These books were produced to meet the needs of comprehensive schools, adapting the SMP O-level course for a wider ability range. They are very widely used for CSE courses, but do not cover the whole of the SMP O-level course.

(vi) SMP Books X, Y, Z These books extend the material of Books A-G to cover the SMP O-level course. Thus, in a comprehensive school, all pupils can begin with the same books and the more able can move on to the O-level course while others use Book H to complete a CSE course.

(vii) SMP Cards I & II In the 1970s individual learning in the first years of secondary schools led to the production of these workcards. They cover the same ground as Books A-D.

(viii) SMP 7-13, Units 1-6 This course was intended to meet the needs of middle schools, for pupils aged 7-13. It is in workcard format, with some small booklets in the later units. Fewer middle schools have been created than was expected and so this course is chiefly used in primary schools (Units 1-4) and, to some extent, in the first two years of secondary schools (Units 5-6).

Other SMP series are:

(ix) SMP Individualised Mathematics This series, which is in the process of publication now, covers the SMP O-level course, but

- the text is presented in a 'programmed' style
- the content is presented by topics.

- (x) SMP Additional Mathematics This course with its examination is used in two ways. Some schools use it with able pupils aged 15 to extend the course presented in Books 1-5. Other schools use it to provide a subsidiary course in the first year of the sixth form for pupils who are not studying A-level mathematics.
- (xi) Supplementary Booklets, 1-5 These contain extra exercises for pupils. There is one booklet for each year of the secondary school up to age 16.
- (xii) SMP Calculator Series These booklets were produced to support the SMP O-level Syllabus C for which the use of an electronic calculator is required.
- (xiii) SMP Topic Mathematics This series provides a one year course for pupils aged 16-17 whose achievement in the CSE examination was between grades 3 and 5, providing the opportunity to improve the grade obtained at age 16.
- (xiv) SMP Computing in Mathematics These publications do not form a course, but are a resource to be used as the teacher decides, intended to encourage the use of computers in the teaching of mathematics.

Work in progress

Besides the rewriting already mentioned above, the Additional Mathematics and Further Mathematics books are being revised.

The major activity, however, is SMP 11-16.

Experience with Books A-H and X, Y, Z has shown that revision is necessary. The language level and the approach are not suitable for many pupils. The mathematical content of the course must be re-examined; some topics have proved impossible to teach in anything but a superficial way to less-able pupils. Teachers in secondary schools, many of whom have only limited mathematical qualifications, need more support in the teaching materials. SMP 11-16 is designed for use by the most able 85% of pupils. Although it draws on earlier SMP writing and courses, there are significant differences and it is better seen as a new series rather than as a revision of old materials.

SMP examinations

The SMP GCE O-level and A-level examinations were important factors in establishing SMP courses in schools and they have continued to influence the teaching of these courses.

SMP O-level Syllabus C was the first to incorporate changes recognising the role of the electronic calculator in changing both content and teaching approaches. A calculator is required in the examination.

SMP GCE examinations are available to any school, but there is no similar arrangement for CSE examinations. Different CSE examining boards have developed their own syllabuses and examinations based on SMP Books A-H.

Work is in progress to establish suitable examinations for pupils who follow the new course, SMP 11-16.

(Editor's note: a booklet of SMP GCE syllabuses was available to participants at the Seminar).

SMP methodology

Three features have characterised all SMP writing:

- (a) Team authorship All books and teaching materials have been produced by a team of authors. In this way a variety of experience and knowledge has been available.
- (b) Authors are schoolteachers With few exceptions SMP authors have been teachers in schools who have continued with their classroom teaching while writing. It has been necessary for some authors and editors to be relieved of their teaching duties to provide sufficient time for work to be completed on time.
- (c) Testing of drafts Each book is produced in draft form and used in pilot schools. The published version is a revision of the draft incorporating improvements suggested by teachers in the pilot schools.

The MEI project: a contrast

The Mathematics in Education and Industry project (MEI) began at about the same time as the SMP. In its early years a number of teachers were seconded to industry to observe the use of mathematics.

The project concentrated its attention on A-level, taking the view that schools could more easily make changes at this stage, where fewer pupils are involved.

A-level syllabuses and examinations were set up, but no textbooks were produced. Schools were left to find or develop their own teaching materials and, although a number of books have now been written with the MEI syllabuses in mind, there are none produced by the project itself. In this respect the approach of MEI resembles conventional practice in English schools. However, one result has been that the number of schools involved in the MEI has remained small. They are mostly independent schools, although a number of maintained schools, especially selective schools, also use the MEI examinations.

The syllabuses include a number of 'modern' topics, and there is a strong emphasis on probability, but the questions set in the examinations have remained more conventional than those in SMP examinations.

The future

The future role of Projects will depend on:

- where initiatives for curriculum change are seen to lie
- the funds available and their sources.

In England we seem to be moving towards a somewhat greater measure of central direction in education. Local education authorities are being pressed to establish a 'core curriculum': a group of subjects to be included in the curriculum of all pupils. The inclusion of mathematics in this core curriculum only makes formal what is already general practice; all pupils study mathematics to age 16. However, the disbanding of the Schools Council and its replacement by two nominated bodies seem to suggest a move to centralisation. The Criteria being developed for mathematics in the new system of examinations at age 16, including a core of content, suggest more intervention from the Department of Education and Science than has been the case in the recent past. If that is the way in which we are moving it may be suggested that initiatives for curriculum change, and the funds for it, should both come from central government funds. Changes of this kind will take time, and independent Projects will continue to have a contribution to make for some time, perhaps to the end of the century.

For the SMP the major problem will be to find sufficient funds to tackle new work on the scale that is necessary. The early pragmatic approach must give way to a more reflective one; ways must be found to use the results of educational research. Most seriously, the scale of new work, attempting to meet the needs of the wide spread of mathematical ability, poses serious problem of resources and methodology.

In his presentation, Mr Hersee clarified the two meanings of the phrase 'project teacher'. Some teachers are involved in the development work of the project; a very much larger number of teachers are in schools using the project's materials.

The question was raised as to whether one can afford the financial disadvantages of producing a variety of courses. Since different courses are developed to meet different needs, both of pupils and of teachers, the variety is not just an accident; it is deliberately planned, in the light of the need for differentiated curricula. Of course it would be cheaper to mass-produce a single course, but in the English view this would have serious disadvantages, such as uncompleted work, pupil's different levels of background understanding when the teacher returned to a topic, and so on.

One advantage of the English system is that one can have small scale experiments and evaluate them. For example, when in 1976 small pocket calculators became available, various experiments were undertaken in different parts of the country. This led to new books and syllabuses. These have proved very successful, and have been used as a basis for further work with a calculator, and have had a major influence on the writing of recent series of texts.

Session 6. How to Achieve what is Planned in Mathematics Education

MATHEMATICS SYLLABUSES IN THE SOVIET SECONDARY SCHOOL

Mrs M A Yermolaeva, Central Schools Board, Ministry of Education of the USSR

(Editor's note. Although Mrs Yermolaeva was not able to attend the Seminar, she had been a member of the Soviet delegation to the first seminar in Oxford, 1981, and her paper was submitted to the British participants along with the other papers for the second Seminar. It is included at this point in the report in view of the clarity with which it describes the Soviet secondary mathematics curriculum. Some of the material overlaps with that in the following paper by Professor Gonchar).

Before discussing the mathematics syllabus for general schools in the USSR, the following factors must be borne in mind:

1. The study of mathematics in secondary schools is compulsory for all children.
2. There is a single mathematics syllabus for general schools and all other types of educational institutions providing secondary education: evening (correspondence) schools, professional-technical schools and special secondary institutions (technical colleges).
3. Successful completion of the secondary school's mathematics syllabus provides the right to enter the universities and other institutions of higher education.
4. Of total teaching time, 57.5 hours a week (ie 1955 hours of total academic time) are devoted to the study of mathematics.

This time is distributed across the year groups as follows:

Hours per week, by classes

	I	II	III	IV	V	VI	VII	VIII	IX	X	Total
mathematics	6	6	6	6	6						30
algebra	-	-	-	-	-	4	$3/4$	4			11/12
algebra & basic calculus	-	-	-	-	-	-	-	-	3	$3/2$	6/5
geometry	-	-	-	-	-	2	$3/2$	2	2	2	11/10

number of hours a
year (34 academic
weeks per year)

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The teaching of mathematics in secondary school is intended to enable pupils to form a correct understanding of the nature of mathematics, of the essence and character of its methods, of the place of mathematics in the system of sciences and its role in science, technology and industry, and to develop spatial awareness and imagination and the skills of logical and creative thinking.

The teacher of mathematics has the right to choose between various methodologies and ways of presenting the syllabus material, including those which increase the pupils' interest in the study of mathematics, help them to master mathematical concepts and methods and provide them with stimulation.

At each stage in the teaching of mathematics the pupils' study of the syllabus material is based on deep and sound knowledge and the skills which they have acquired in the course of the previous year's work. With this in mind, academic time is set aside specially for the consolidation of earlier material and for exercises and problem-solving. Throughout the academic year there is in every class a revision of what has been studied, and this helps towards a deeper and more sound mastery of the material covered, systematizes it and reveals interconnections, similarities with and differences from the new material.

In order to develop pupils' interest in the study of mathematics and to raise the level of their 'mathematical culture' it is recommended that extra-curricula activities should be carried out systematically (clubs, evening activities, quizzes, olympiads and so on). The same purpose is served by optional courses which are introduced from the 7th class onwards.

In lessons and extra-curricula activities pupils are acquainted with facts relating to the history of mathematics, the lives and work of famous mathematicians and the achievements of the Soviet school of mathematics. Serious attention is given to the development of the mathematical capabilities of those pupils who show particular interest in the study of mathematics.

The basic mathematical content of the syllabus for classes I-III (age 7-9) comprises natural numbers and operations on them.

In the primary mathematics course the study of natural numbers and the arithmetical operations performed on them is organised concentrically. The syllabus proposes a system whereby the domain of the numbers being considered is gradually broadened (tens - hundreds - thousands - multi-digit), and at the same time as the domain of number is being broadened, so the childrens' knowledge of the natural number series, which they acquired in the earlier stages of their course, is being deepened, systematized and generalized.

In the mathematics course for classes I-III four operations with natural numbers are studied: addition, subtraction, multiplication and division. The study of these operations is related first of all to the necessity of giving the pupils a sound knowledge of the skills of calculation. For each operation various problems are considered, in which the operation in question is applied and which reflect particular real-life situations.

The syllabus aims to acquaint children with some of the properties of the arithmetical operations. Their use helps to make the study of calculating methods more conscious, which is important not only for the inculcation of sound calculating skills but also for the pupils' general development.

From the first class children are acquainted with separate examples of magnitude (length, mass, volume etc) the units of measurement and the correlations between these units. First of all the children learn about length, in connection with the study of the concept of the line-segment and measurement of it. Subsequently the study of magnitude is linked as far as possible with the study of the concept of number.

The mathematics course for classes IV-V (age 10-11) aims to generalise and develop, on the basis of new material, the mathematical knowledge and skills which the pupils have acquired in the primary school, and to prepare them for the systematic courses in algebra and geometry. During the course pupils acquire the skills of calculation with natural and

whole numbers, fractions and decimals; they gain the notion of the use of letter symbols for the writing down of numbers and learn to make up (within the terms of a problem) and solve simple linear equations; they continue to work with geometrical concepts, they learn to identify and reproduce more complex geometrical figures, and they acquire the skills of measuring geometric magnitude.

In 8-year school (classes VI-VIII - age 12-15) two subjects are studied - algebra and geometry.

Through the study of the algebra course it is intended that pupils should:

- generalise and systematize their knowledge of real numbers; develop and consolidate their skills of calculation;
- master the skills of transposition of the basic types of algebraic expression (multinomials, algebraic fractions, powers and roots);
- master the ways of solving algebraic equations and inequalities of the first and second degree and equations and systems reducible to these;
- study the simplest elementary functions and their properties.

Mastery of the skills of transposition presuppose a sound grasp of the basic algorithms (expanding brackets and inserting brackets, taking the common factor outside brackets, reduction of like terms, addition, subtraction and multiplication of multinomials, use of the formulae for short multiplication, arithmetic operations on algebraic fractions in simple cases, transformation of powers and roots).

The study of functions in 8-year school is primarily intended to facilitate mastery of the ways of exploring functions and the construction of the graphs of functions by elementary methods on the basis of relatively limited functional material (direct and reciprocal proportional dependence, linear and quadratic functions, the functions $y = x^n$ and $y = \sqrt[n]{x}$ where n is a natural number.)

The geometry course aims at a systematic study of the basic facts of plane geometry and the methods it uses, which is based upon the level of geometric preparation reached by pupils in their previous classes and which must then in turn form the basis of the study of related subjects and solid geometry in classes IX-X.

During this course pupils are supposed to:

- acquire systematic knowledge of the basic figures in a plane and of their most important properties;
- gain a notion of the equality and similarity of figures, of the basic types of geometrical transformation and their applications in geometry;
- acquire the skills of geometric construction which are essential for carrying out graphical work (which occurs frequently), and also the skills of measuring and calculating lengths, angles and areas, which are used in the solution of various geometrical and practical problems;
- become acquainted with the use of analytical apparatus (algebraic transformations and equations, elements of trigonometry, analytic geometry and vector algebra) for the solution of geometric problems.

As a result of completing the geometry course pupils should have an active knowledge of geometrical concepts, the basic properties of the figures they have studied and the methods employed in geometry; they should be able to reproduce proofs of the basic theorems of the course and independently solve typical problems and write down their solutions. They should be able to draw conclusions on the basis of statements about the equality and similarity of figures, to construct (with evidence of the course of reasoning) the figure which is obtained when a particular geometric transformation is performed on a given figure; to find in concrete situations (which do not require any artificial additions) equal (similar) triangles, to be able to prove their equality (similarity) on the basis of corresponding features or the properties of geometrical transformations.

Pupils must have developed both the skills of applying the analytical apparatus they have studied to the solution of standard problems with the help of a particular algorithm (the graphical addition and subtraction of vectors, the solution of triangles, the use of the scalar product to calculate the length of a vector and the angle between vectors, etc), and also to the solution of simple practical problems which are reducible to standard problems (measuring area, etc.)

In classes IX-X (age 16-17) algebra and basic calculus and geometry are studied.

In the algebra and basic calculus course the study of the major questions of algebra, which was begun in the 8-year school, is completed, and some of the elementary concepts of mathematical analysis (elements of differential and integral calculus) are taught as being the apparatus for the solution of geometric, physical and other applied problems.

In the algebra and basic calculus course the class of functions studied is broadened to include functions with powers, exponential, logarithmic and trigonometric functions, and

knowledge of the basic elementary functions and their properties is systematized and supplemented, while at the same time the apparatus of analysis is introduced. Much time is devoted to the formation of the skills of transposition of exponential, logarithmic and trigonometric expressions and the use of these transpositions for the solution of problems (the solution of equations, inequalities and simultaneous inequalities, finding derivatives, the solution of problems from geometry by the use of trigonometry).

In the study of the elements of mathematical analysis the main emphasis is on the pupils' understanding of the geometrical and physical significance of the derivative and integral and on their ability to use the methods of differential and integral calculus for the solution of applied problems.

The solution of simple transcendental equations, inequalities and simultaneous inequalities made up of functions which are studied at school, serves the purpose of broadening and systematizing the knowledge which has already been acquired and also of working through the technical skills required for dealing with basic elementary functions.

The aims of the geometry course for classes IX-X are the systematic study of elements of solid geometry, which are essential for the solution of frequently - met practical problems, and also the development and consolidation of the main "threads" which were first taken up in the 8-year school.

Through their study of the course pupils should:

- acquire systematic knowledge of the basic types of spatial figures and their properties;
- learn to use the theorems from the course for the solution of problems involving the representation of spatial figures in the plane and the finding of magnitudes (angles, lengths, areas and volumes);
- become acquainted with analytical methods for the solution of problems of solid geometry.

The course for classes IX-X calls for a higher degree of logical rigour than does the 8 year school. At this stage the notions of equality and similarity of figures and the methods of geometrical transformations (with which the pupil is familiar from the course of plane geometry) are consolidated. Pupils must be able freely to make use of the knowledge acquired from the plane-geometry course to solve problems of solid geometry; they must have mastered the concepts and be able to prove the theorems from the course. When presenting their solutions in written form pupils must show with sufficient detail that they are soundly based.

The line of geometrical constructions begun in 8-year school takes on a qualitatively new character in the senior forms. Pupils are expected to be able to represent spatial figures in the plane on the basis of the properties of parallel projection; they must be capable (with the help of models, drawings and in their heads) of conceiving the relationship between geometrical figures and of finding solutions on this basis and interpreting results; they must be able to solve simple problems involving the construction of projections of spatial figures and also problems involving the construction of sections. Elements of spatial imagination and constructional skills are also worked through in the course of solving problems involving "imaginary constructions" (such as constructing a perpendicular to a plane). Pupils must be able to solve simple problems involving the finding of lengths, angles, areas and volumes (including problems with practical content). Here it is necessary that the pupil should be able both to translate the practical problem into the language of geometry and to solve it in a way which is sufficiently fully shown to be well-grounded.

The level of concrete demands which are made on pupils' knowledge and skills throughout the study of the various branches of mathematics is basically determined by the level of difficulty of the exercises contained in the text-book for the appropriate class. This same level of difficulty is matched by the content of the two examinations which all pupils take on completing 8-year and secondary school.

REQUIREMENTS FOR THE MATHEMATICAL TRAINING OF SCHOOL PUPILS

Professor A A Gonchar, Mathematical Institute, Academy of Sciences of the USSR

General requirements are: obligatory mastery of the basic concepts, results and methods set out in the syllabus on a level which is sufficient for the study of the course material and related disciplines. Pupils must have a knowledge of the theoretical content of the mathematics course and solve the problems which it sets them.

Mathematics. (Classes 1 to 3)

The course involves the study of the arithmetical operations on natural numbers and their application to simple magnitudes, visual acquaintance with individual geometrical figures and their properties. The course material must lay the practical foundations for the subsequent study of mathematics.

During the course pupils are expected to:

- a) Learn to carry out the arithmetical operations on natural numbers.

Here it is necessary to develop sound skills in the written performance of the addition, subtraction and multiplication of two- and three-digit numbers and also the division of such numbers by one- and two-digit numbers, which presupposes knowledge by heart of addition and multiplication tables for one-digit numbers. The formation of the skills of written calculation and in the simplest cases of mental calculation must be raised to the level of fluent and mistake-free performance of calculations.

- b) Learn to measure simple magnitudes and carry out corresponding operations on them.

The greatest attention must be focussed on the development of sound skills of measuring magnitude, and mastery of the most widely used units of measurement. Pupils must grasp the simplest dependences between magnitudes and use their knowledge of these dependences in the solution of word problems.

- c) Learn to recognise and reproduce the simplest geometric figures.

Pupils are not dealing with abstract concepts but with real representations of geometric figures and they must learn easily to recognise them in drawings, models and surrounding objects, and master simple drawing skills.

Mathematics. (Classes 4 to 5)

The course involves the generalisation and development, on the basis of new material, of the mathematical knowledge, abilities and skills acquired by pupils in primary school and the carrying out of propaedeutic teaching in order to prepare pupils for the study of the systematic courses in algebra and geometry.

During the course pupils are expected to:

- a) Master the skills of calculation with natural numbers and whole numbers and with ordinary and decimal fractions.

Here the algorithms of calculation with two- and three-digit numbers must be worked through with pupils to the point where they become automatic; pupils must easily carry out in their heads arithmetical operations of a complexity corresponding to that of the given examples, multiplying a two-digit number by a one-digit number and adding two fractions in simple cases. All calculations must be sufficiently fluent; their use in the performance of more complex calculations (for example, reduction to the lowest common denominator, calculation of the value of numerical expressions etc.) must not cause a pupil any difficulties.

- b) Acquire the idea of the use of letter-symbols for the writing down of the properties of numbers and learn to make up and solve simple linear equations within the terms of problems

Pupils must understand the meaning of the writing down in letters of the properties of arithmetical operations; they must be able to make up expressions consisting of numbers and letters within the terms of a problem and carry out simple transformations of expressions; and on this basis to make up linear equations and solve them with the help of algebraic methods.

- c) Continue their acquaintance with geometric concepts, learn to recognise and reproduce more complicated geometric figures and acquire the skills of measuring geometric magnitude.

Here it is necessary to bear in mind the propaedeutic nature of the study of geometry in classes 4 to 5, so that the purpose is not that pupils should acquire systematised knowledge about abstract geometric figures but that they should become acquainted with their real representations. Pupils must be able to carry out simple measurements and constructions with the help of suitable instruments and solve simple problems involving finding lengths, areas and volumes.

Algebra. (Classes 6 to 8)

The course involves the systematic study of the elements of algebra and elementary functions based on the level of calculating ability acquired by pupils in previous classes with the intention of supplying the necessary basis for the study of related subjects and the propaedeutics of the study of algebra and basic calculus in classes 9 to 10.

During the course pupils are expected to:

- a) Generalise and systematise their knowledge of real numbers and develop and consolidate their skills of calculation.

Pupils must acquire an idea of the representation of real numbers using recurring and non-recurring decimals. Here it is important to sustain concentration on the maintaining of a sufficiently high level of calculating technique and the raising of pupils 'calculatory culture' (the planning of calculations, their organisation, the use of approximate calculations). This must be achieved by means of the consistent increase in the proportion of calculation in the study of the basic course material. Pupils' calculating skills must be further developed in the study of questions relating to approximate calculations, where, as well as further work on the algorithms of calculation, formation of the skills of evaluating the results of calculations must take place. As pupils master the algorithms of calculation and increase the volume of their knowledge about numerical functions, so the volume and complexity of calculations significantly increases, requiring the introduction of tables and mathematical instruments (slide-rule and later the calculator).

- b) Master the skills of transposition of the basic types of algebraic expression (multinomials, algebraic fractions, powers and roots).

The basic algorithms (expanding brackets and factorising, taking the common factor out of brackets, reduction of similar terms, addition, subtraction and multiplication of multinomials, the use of formulae of short multiplication, arithmetic operations on algebraic fractions in simple cases, transformations of powers and roots) must be soundly grasped. Pupils must not get into difficulties in the carrying out of corresponding transformations requiring use of the algorithms indicated in the course of problem-solving.

- c) Master the methods of solving algebraic equations and inequalities of the first and second degree and equalities and systems reducible to them.

The most important aim is the sound mastery of the algorithms for solving the basic types of equations, inequalities and systems. Mastery of these algorithms must be accompanied by visual geometric illustration (graphical solution of equations, use of graphs for analysis and examination of solutions, etc). Pupils must acquire certain skills in the reduction of more complex equations and inequalities of basic type. In the study of simultaneous equations attention must be focussed on the working through of practical methods for their solution.

- d) Study simple elementary functions and their properties.

The main purpose of the study of functions in eight-year school is mastery of the methods of investigating functions and constructing their graphs using elementary methods and on the basis of relatively little functional material (direct and inverse proportional dependence, linear and quadratic functions and the functions $y = x^n$ and $y = \sqrt[n]{x}$ where n is a natural number). It is also important to develop, on the basis of sufficiently simple examples, the ability to express in functional form relationships between magnitudes and to use with ease various methods of representing functions, making the transition from one of them to the other (calculation of values of functions given by a formula, reading graphs, etc). As a result the prerequisites for the confident use of these methods in the study of algebra and basic calculus in classes 9 and 10 must be created.

Geometry (Classes 6 to 8)

The course involves the systematic study of the basic facts of plane geometry and its methods, starting at the level of geometric preparation which pupils have attained in previous classes with the intention of providing the necessary basis for the study of related subjects and solid geometry in classes 9 and 10.

During the course pupils are expected to:

- a) Acquire systematic knowledge of the basic figures in the plane and their important properties.

As a result of following the geometry course pupils must have an active mastery of geometric concepts; know the basic properties of the figures they have studied and

the methods which geometry uses; reproduce proofs of the basic theorems in the course, be capable of independently solving typical problems and writing down their solutions. Considerable importance is attached to the development of geometric intuition: drawings must become an important heuristic device, facilitating the formulation and checking of hypotheses and the selection of ways of solving problems. The combination of the visual quality with rigour must be an integral part of geometric knowledge. Pupils must also be able to apply the knowledge they have acquired to the solution of simple practical problems.

b) Acquire an idea of the equality and similarity of figures and of the basic types of geometric transformations and their uses in geometry.

Pupils must be able to deduce the consequences of statements about the equality and similarity of figures; to construct (adducing proof) the figure which results when a given figure undergoes a particular geometric transformation; to find in concrete situations (which do not require artificial additional constructions) equal (similar) triangles, be able to prove their equality (similarity) using appropriate indications or the properties of geometric transformations. These abilities must be raised to the level where pupils are able independently to solve problems requiring the use of the afore-mentioned methods.

c) Acquire the skills of geometric construction which are necessary for the large amount of graph-work, and also the skills of measuring and calculating lengths, angles and areas, which are used in the solution of a variety of geometrical and practical problems.

Here the pupils must carry out with ease the constructions laid down by the syllabus and solve simple problems involving construction by means of ruler and compasses. Skill in the use of instruments of measurement is raised to the level necessary for the carrying out of measurements of a practical nature and sufficient for the use of corresponding skills in related subjects. Pupils must also solve simple problems involving the finding of magnitude without resorting to the direct use of a single formula or theorem.

d) Become acquainted with the use of analytical methods (algebraic transformations and equations, elements of trigonometry, analytic geometry and vector algebra) for the solution of geometrical problems.

Pupils must develop skills in the use of the analytical methods which they have studied for solving standard problems whose solution is carried out with the help of a particular algorithm (graphical addition and subtraction of vectors, solution of triangles, use of scalar product for calculation of the length of a vector and the angle between vectors etc), and also for solving simple practical problems which can be reduced to standard ones. Pupils must become acquainted with the use of the methods they have studied for solving geometrical problems.

Algebra and Basic Calculus (Classes 9 to 10)

The aims of the algebra and basic analysis course are to bring to a conclusion the various threads of content in the study of algebra which were first taken up in eight-year school and to acquaint pupils with the basics of mathematical analysis (the elements of differential and integral calculus) as apparatus for the investigation of functions and the solution of geometric, physical and other applied problems. At this stage, when pupils' secondary training is nearing completion, the course aims to generalise and systematise the knowledge and to consolidate the abilities and skills which pupils have acquired through their study of the algebra course in eight-year school; these aims are realised on the basis of planned revision.

During the course pupils are expected to:

- a) Broaden the scope of their knowledge of numbers by acquaintance with complex numbers and operations on them.

Pupils must learn to carry out operations on complex numbers in algebraic and trigonometric form, making the transition from one form to the other; to represent complex numbers geometrically and to solve with ease quadratic equations with negative discriminants.

- b) Increase the range of functions studied by the inclusion of functions with powers, exponential, logarithmic and trigonometric functions and systematise and supplement their knowledge of the basic elementary functions and their properties

In the algebra and basic calculus course elementary functions must be the subject of systematic study both on an elementary level and by means of the use of the apparatus of analysis. This will considerably increase the polytechnical potential of

the school mathematics course. The development of the function theme ends with the study of elements of differential calculus.

- c) Master the skills of transposition of exponential, logarithmic and trigonometrical expressions and their use in the solution of problems.

Pupils must acquire an easy mastery of the basic identities on a level which will allow them to perform such transformations with sufficient confidence in the solution of various types of problems (solution of equations, inequalities and systems, finding derivatives, solutions of problems of geometry through the application of trigonometry).

- d) Become acquainted with the basic concepts, results and methods of mathematical analysis in sufficient breadth to investigate elementary functions and solve simple geometric, physical and other applied problems.

When the elements of mathematical analysis are being studied in school it is not the technical or formal aspect which should be singled out but the content, with the emphasis first and foremost on pupils' understanding of the geometric and physical meaning of such concepts as derivative and integral, and also on their understanding of the possibilities provided by the application of methods of differential and integral calculus to the solution of a large group of important applied problems. Purely technical skills should not be developed more than is necessary for the achievement of these ends.

- e) Learn to solve simple transcendental equations, inequalities and their systems constructed with use of functions which have been studied in school.

The study of this traditional material is intended to broaden and systematise knowledge which has already been acquired relating to the solution of equations, inequalities and simultaneous inequalities and also to assist the working through of technical skills for dealing with the basic elementary functions studied in school.

Geometry (Classes 9 to 10)

The course involves the systematic study of elements of solid geometry which are essential for the solution of frequently encountered practical problems and the development and consolidation during the course of those main threads first taken up in

eight-year school. As secondary school teaching is reaching its conclusion at this stage great attention must be devoted to the consolidation of basic skills and the systematisation of knowledge acquired, through the study of new material as well as through planned revision.

During the course the pupil is expected to:

- a) Acquire a systematic knowledge of the basic types of spatial figures and their properties.

Taking into account pupils' familiarity with geometry acquired in eight-year school, greater demands are made in classes 9 to 10 as regards the level of logical rigour than was previously the case. In the course of teaching it is necessary to consolidate the ideas of equality and similarity of figures and the methods of geometric transformations which are already familiar from the plane geometry course. Pupils must be able to apply with ease the knowledge they acquire in their plane geometry course to the solution of the problems of solid geometry; they must master the concepts and prove the theorems from the course. In their written formulation of solutions pupils must give reasonably full justifications.

- b) Learn to use the theorems of the course for solving problems involving the representation of spatial figures in a plane and the finding of magnitudes (angles, lengths, areas and volumes).

The theme of geometrical constructions begun in eight-year school takes on a qualitatively new character in the senior forms. Pupils are expected to be able to represent spatial figures in the plane on the basis of the properties of parallel projection; they must be capable (with the help of models, drawings and in their heads) of conceiving the relationship between spatial figures and of finding solutions on this basis and interpreting the results. In this connection it is important that pupils should be able to solve simple problems involving the construction of projections of spatial figures and also problems involving the constructions of sections. Elements of spatial imagination and constructional skills are also worked through in the course of solving problems involving 'imaginary constructions' (such as constructing a perpendicular to a plane). Pupils must be able not only to reduce a solution to a numerical result but also to translate a practical problem into the language of geometry and solve it in a way which is shown to be sufficiently well grounded.

c) Become acquainted with the analytical method of solving the problems of solid geometry.

Pupils must consolidate the appropriate skills in the solving of standard problems (which they acquired in Classes 6 to 8) using solid-geometry material, and use analytical methods with confidence in the solution of geometrical problems.

In his presentation, Professor Gonchar explained that his paper was based on a document prepared by a committee of the Academy of Pedagogical Sciences, assisted by a large number of other people. This document was concerned with the requirements of content, and had nothing to do with methods of teaching. It set out the necessary and sufficient content of knowledge for all pupils at secondary school.

The former "Kolmogorov" syllabus, introduced in the 1970's, was based on sets and structure. In practice it had been found to be too formalised. Structural ideas such as set, relation and function had to be introduced using non-mathematical examples, which took a lot of time which could have been used for pure mathematical work. For example, complex numbers had had to be excluded. Furthermore, complicated symbolism was being used for essentially simple ideas. The basic principle is that the teaching of mathematics should present a systematic body of knowledge properly.

The following main points arose in discussion.

1. The inclusion of complex numbers in the syllabus for Classes 9-10 is controversial. The intended new syllabus restores them.
2. Examples were asked for to indicate the kinds of pupil tasks implied by certain elements in the syllabus. The following were given:
 - a. "Mastery of the units of measurement". Knowledge of names of all the metric units, how many cm in a metre, converting km per hr to metres per sec, finding area of rectangle with sides given in cm.
 - b. "Mental calculation of 2 digits x 1 digit, eg 52×7 ". Any method would be accepted but the taught method would be 50×7 , then 2×7 .
 - c. "Applications of calculus to geometrical, physical and other applied problems". Problems of area, volume, speed, maxima/minima, including eg finding volume of solids of revolution of curves such as $y = x^2 + \frac{1}{x^3}$ between given limits.

3. Calculus ideas and notation are used in the later stages of the physics course in the study of speed and distance, though not when these concepts are first introduced.

4. Both slide rules and calculators must be learned, but the phasing remains unclear. Many engineers still use a slide rule so it is necessary to teach it. It will probably disappear soon. Similarly decimal logarithms will not need to be introduced so early.

5. Although the abacus is used in shops very efficiently, Professor Gonchar thought it better not to teach its use in school.

6. Probability and statistics were considered by the committee, and should be included in the next syllabus revision. This will involve a big change in teaching methods, because they should be taught not as mathematics but more like science. The level of formalism which is traditional for the mathematics course is unsuitable for approximate calculation and for probability and statistics (and possibly for calculus as well).

Mention was made of the successful Nuffield guide on Probability for 11-12 year olds, with a very practical treatment.

PLANNING THE MATHEMATICS CURRICULUM FOR DESIRED OUTCOMES

Douglas Quadling, Cambridge Institute of Education

One of the most obvious differences between the two groups of participants in these seminars is the organisational context within which decisions about the mathematics curriculum are taken. In England almost the only requirement laid down by central government is that children should receive full-time education between the ages of 5 and 16, for a certain minimum number of days in the year. More detailed decisions about the curriculum within the public education system are the responsibility of local government (units covering populations of about half-a-million inhabitants on average); many of these decisions are delegated in practice to schools, although advice may be offered by the local administration particularly at the level of primary education (age 5-11). It is not even a requirement that a school should provide instruction in mathematics, or that a child should learn mathematics at any stage of his education. And although as a matter of fact mathematics is taught in all schools, its teaching may be organised in a wide variety of ways (including the possibility of integrating the teaching of mathematics for some pupils with other subjects, either wholly or in part) and there can be large differences in content and timing from one school to another. By contrast, we understand that in the Soviet Union central government lays down not only the number of hours to be devoted to mathematics at each stage of a child's education, but also the content of the course on a year-by-year basis, some of the methods to be used in presenting this content (eg order of treatment, whether a particular proof should be learnt) and the textbook through which the material should be taught.

It might seem that these differences would be a barrier to effective common discussion of questions concerning the curriculum, but it is suggested that on the contrary it could be a source of strength to the seminar. Whether the unit of decision and implementation is a single school or a large state, it is equally necessary to identify desired outcomes and consciously to plan the curriculum towards their achievement. However, the great differences in scale between the two systems will enable us to bring very different sets of insights to the discussion. In the Soviet Union it is possible to subject proposed innovations to large-scale experiment, to engage the best experts in curriculum construction, to set up programmes of teacher training directly related to the school curriculum. On the other hand the mathematics department of an English school can observe at first hand the effect of their decisions on children's learning of mathematics, maintain contact with their colleagues who teach related subjects such as physics or economics, incorporate innovations (eg the use of calculators) quickly, and design a curriculum which takes

account of local conditions (eg a predominant form of employment, or a seaside environment). It is to be hoped that the discussion of this theme can therefore be concentrated on the common problems, but enriched by the variety of experience which our respective delegations can contribute to their solution.

Desired outcomes: general categorisation

Education in school has both general and specific purposes, and both personal and public functions.

It may be helpful to refer briefly to certain general desired outcomes of the educational process, such as:

recognition of the mutual obligations of the individual and society,
the capacity for constructive cooperation within a group,
the ability to work independently and to use appropriate resources,
the development of manual dexterity,
the ability to listen and the power to communicate,
etc, etc.

Mathematical education has its part to play in furthering all these aims, and this has implications for the way in which mathematics is taught and assessed, and possibly even to some degree for the content of the course. (For example, the study of statistics should have social as well as mathematical consequences.)

For the most part, however, it will probably be profitable to concentrate in the seminar on specifically mathematical outcomes. Before coming to the detail of the curriculum, these will be categorised in general terms. Two ways of looking at them are suggested: in terms of the needs of the individual, and the needs of society.

(A) Desired outcomes for the individual

(A1) Each pupil needs to acquire sufficient confidence and understanding in dealing with number (and other mathematical ideas) to enable him to follow his chosen life-style (including incidental uses at work and in his leisure pursuits) without hindrance.

There are several points to make about this. First, it does not represent the same body of mathematical knowledge, or the same level of competence, for everybody. Second, it does

not require us to prescribe particular ways of carrying out mathematical processes; any method will do, provided that it is reasonably efficient and generalises sufficiently to meet the needs of the individual. Third, although it is customary to talk of educating children to enter the adult world, it must be recognised that children also have such needs for mathematics, so that this outcome should have high priority in the early years of school.

(A2) Pupils need to have sufficient command of the mathematics which they will use in studying other subjects in the school curriculum such as science, geography, craft and economics.

This is a difficult criterion for syllabus planning, since it involves collaboration with other groups whose educational priorities may be expressed differently from our own. Moreover, this is a process which needs to be repeated each time that changes are made in the curriculum for any of the subjects concerned. There may be a clash of interests in deciding what is appropriate at any given stage, and what is implied by the word "command" in the above statement. For some children, learning may be more effective if a mathematical concept (eg derivative) is first discussed and practised in a context (eg physics) to which it can be seen to be applicable.

(A3) By the time that he leaves school, each pupil should either know as thoroughly as possible the specific mathematics he will use in his future work or have the mathematical background needed to follow a course of training to initiate him into that work.

This is another difficult area because of the variety in the mathematical demands of different jobs. Whilst there are some forms of mathematical reasoning which are needed in so many fields as to justify teaching them to all children (eg proportion, or use of a calculator), other topics (such as trigonometry) will be of great importance for a few pupils in a class but irrelevant for the majority.

Note that (A2) and (A3), although similar in some respects, play different roles in curriculum planning, since (A3) relates only to the knowledge acquired before a pupil leaves school, whereas (A2) has implications for the timing and sequencing of topics right through the period of schooling. This suggests some kind of allocation of priorities to the outcomes (A1-3) as indicated by Fig.1.

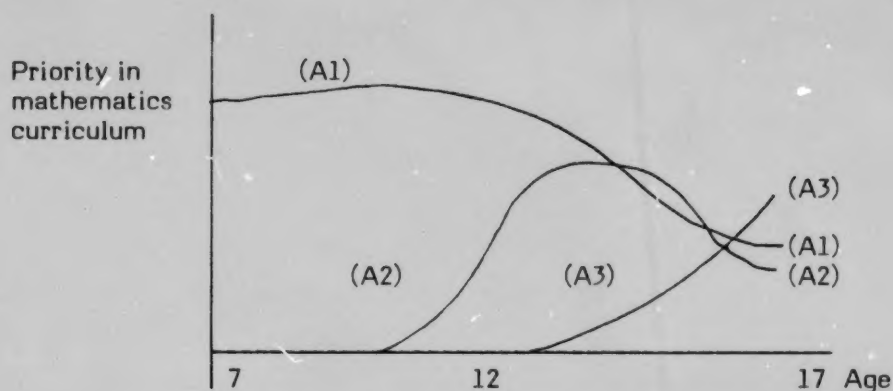


Figure 1

(A4) Pupils should be aware of the existence and fascination of numerical and spatial relationships, of mathematics as a human activity with internal and external motivations, and of the processes and forms of reasoning through which mathematics is developed.

There are two aspects of this - initiation into a cultural heritage and fulfilling an intellectual need - but it could be argued that for many pupils "appreciation" is a more important outcome than "competence". It has been deliberately omitted from Fig. 1, since there is a wide disagreement about the priority that should be given to this outcome in school mathematics, and whether this priority is the same for all pupils. This is a question on which it might be interesting to have an exchange of views in the seminar. There is, however, an interaction of this outcome with (A1-3): if children have some understanding of the role of mathematics and the nature of mathematical thinking, then the other outcomes may be more successfully realised. This is a hypothesis not easily tested, but one which has important implications for curriculum development.

It also needs to be recognised that whereas (A1-3) have rather clear implications for the content of the mathematics syllabus, (A4) can be realised by means of many alternative topics within mathematics.

(B) Desired outcomes for society

(B1) The state needs to ensure that its population is adequately equipped, mathematically as in other respects, to respond to the requirements of the national administrative organisation.

(B2) A school needs to ensure that its overall curricular objectives are not unreasonably frustrated by a lack of preparation and cooperation in any subject area; in particular, the mathematics teaching must provide adequate support for the curriculum as a whole.

(B3) The state relies on its education system to reinforce it with sufficient expertise, including mathematical expertise, to enable it to develop economically and technologically. This has implications both for the actual knowledge and skills which children acquire in school, and also for their attitudes and receptiveness to subsequent training for future tasks.

(B4) It is in the interests of the universal society that mathematics should continue to develop as an active component of the intellectual debate, both within its own boundaries and in its relationships with other disciplines.

These (B) outcomes are clearly related bijectively to the corresponding (A) outcomes. The difference lies largely in one's point of view. English experience is that mathematics teachers, parents of pupils at school and specialists in mathematical education often express themselves in (A) terms, whereas politicians, industrialists and university professors tend to use the language of (B). Nevertheless, the distinction has been important in England in the conduct of the national educational debate, especially in a time of restricted economic resources.

The contrast between the two is highlighted by the discussion of two fundamental and related problems with which curriculum planners have to come to terms:

- (1) Should there be differentiation within the curriculum to take account of the different levels of attainment of the pupils?
- (2) Should a minimum curriculum be laid down (either for the course as a whole, or for various subsets of it) as a requirement for all pupils?

Differentiation within the curriculum

It is probably best to exclude from this discussion the small number of children, including those designated "mentally handicapped", who appear to derive no benefit from receiving their education in ordinary school classes. We were informed in the first seminar (see p.8 of the English version of the official report) that the Soviet Union has special schools for such children; similar provision exists in England, although the national policy is that, where possible, they should be educated in ordinary comprehensive schools, which often provide for their special needs in "remedial classes". The needs of these children are so exceptional that they must be excluded from the normal process of curriculum planning.

At the other end of the scale, the Soviet contributors to the first seminar described various means by which the education system makes provision for pupils who display a special talent or interest for mathematics - enhanced mathematical training in specially designated schools or through the mathematics correspondence school, the publication of a journal for students, and programmes of optional studies (pp 79-80 of the English version of the report). No comparable national provision exists in England, but there are many local initiatives in the form of special conferences, mathematics competitions, magazines, etc. which have the aim of encouraging children's mathematical interests.

However, the question of a differentiated curriculum is not limited to catering for the needs of the highly gifted and the handicapped. It is of interest to set side-by-side the following two quotations:

"Unless the great differences which exist between pupils are recognised in the provision which is made, those whose potential is high will be denied the opportunity to make the progress of which they are capable and those whose attainment is limited are liable to experience continuing and dispiriting failure." (From the report of the Cockcroft Committee, p 128, published in January 1982).

"The effectiveness of the instructional process ... depends in large measure on successful mastery of an educational content identical and compulsory for all members of society ... The inclusion of optional studies in the teaching curriculum is a further development of the fundamental ideas and principles of Soviet general-educational schools ... The general-educational functions of optional studies consist in offering opportunities to students who show an interest in or an aptitude for a subject to acquire additional knowledge, abilities and skills in that subject". (From the paper by V M Monakhov to the first seminar, p 79 of the English version of the report.)

It appears therefore that the two delegations might have a common interest in exploring further the scope and nature of differentiation within the mathematics curriculum. In order to clarify the situation as it applied to English schools, it may be helpful to add some remarks about the organisation of teaching.

First, it must be pointed out that English schools do not normally follow the practice found in some other countries, of testing all children at the end of each school year and requiring those who fail to reach a prescribed standard to repeat the year's work. This is a system which achieves discrimination between pupils of differing attainment by having

class-groups which are not homogeneous in age. In England, normally all children move up through the school with their year-group; the consequence is that the children in any one year-group constitute a population with a wide variation of mathematical attainment.

Amongst the methods which have been used to make allowance for this variation are:

- (1) To lay down the same basic syllabus for all pupils within a given school year, but to provide enrichment either
 - (i) topic by topic in the shape of harder problems, additional applications, alternative methods, generalisations, etc, for the more successful pupils, or
 - (ii) in the form of additional topics which are not part of the mainstream course.
- (2) To arrange the curriculum for the complete course in such a way that different pupils can work through it at different speeds, so that syllabus content is not tied to a particular year of the course. (This implies that there is some work at the end of the course which is only reached by the more successful pupils).

These alternatives are not, of course, mutually exclusive. For example, it is possible to adopt (2) but also to insert additional material into the coverage of individual topics within the course, as in (1)(i).

One factor which influences decisions about the choice of method is the flexibility available within a school's administrative structure to implement the differentiated curriculum. Two ways in which this is achieved in English schools are:

- (1) To arrange the classes ("sets") within a given year-group so that pupils of comparable attainment in mathematics are placed in the same class.
- (2) To organise the teaching materials in such a way that they can be used by pupils on an individual basis, so that at any given time there is a diversity of work being done by different pupils within the class.

However, it is rare in England to find mathematics available additionally on a basis of optional studies, on the Soviet pattern.

For the purposes of the seminar it may be useful to discuss:

- (1) If the idea of a differentiated curriculum in mathematics is acceptable in principle, are there age-levels at which it is especially appropriate or inappropriate? (It is noted, for example, that in the USSR optional studies are only available from the 7th grade up.)
- (3) Are there some topics in mathematics which are essentially age-related (that is, it is best if all children meet them at a certain specified age) and others which are attainment-related (that is, children should meet them when they have attained a certain proficiency in the subject)?

Specification of minimum content

It might appear to be self-evident that if the working of the state (outcome B1)) or of the school curriculum (outcome (B2)) call for a certain minimal level of competence in mathematics, then the mathematics curriculum for all pupils must be designed so as to try to ensure that this level is reached. Teachers, however, aware of the difficulty experienced by some pupils in achieving and retaining any level of mathematical knowledge, however elementary, are reluctant to commit themselves to such a minimum syllabus, they may well bring to light more basic misunderstandings, to which they consider that higher priority should be given.

Certainly it is unreasonable to require that all pupils should attain mastery of any list of mathematical topics, however small; the most that can be expected is that the pupils should be exposed to this content, and that the teacher should give them every encouragement to master it. Thus the recent report of the Cockcroft Committee, in its preamble to a suggested "foundation list" of mathematical topics, states:

"(We) suggest a list of topics which we expect to be within the capability of most pupils and which we believe that all pupils should attempt so far as they can. We would not wish that any section of the list should be omitted completely, even for those whose attainment is very low, though it must be accepted that such pupils may not be able to tackle all the items in each section." (para 456, p 135).

The Committee's list is very elementary in content, and far smaller than the common syllabus in mathematics laid down in many countries; the view was taken that this had to be the case if it were to be regarded as a realistic target for all pupils at the school-

leaving stage. The report stresses that it should not be regarded as defining the total mathematical experience of pupils of this age, even of those of low attainment. A summary of the list is attached to this paper as an Appendix; complete copies (pp 135-140 of the report) were made available at the seminar.

The pedagogical triangle: desired outcomes, curriculum and children

No curriculum, however good, can of itself ensure that a desired learning outcome is achieved. Even when mediated through the finest teachers, and realised in the best textbooks (and possibly other materials such as television or practical apparatus), it is still possible that some children will be unsuccessful. Thus in designing a curriculum we need to pay attention not only to desired outcomes but also to the ways in which a child's knowledge of mathematics develops.

This is, of course, a highly complex process, and various models have been devised as aids to its interpretation. For younger children, the most successful of these probably remains that originally postulated by Piaget. However, this is a rather general model of development with implications for several areas of the school curriculum; as children mature and mathematics emerges as a distinctive strand of knowledge, it may be useful to identify stages of development specific to the subject. Work in England along these lines has been concentrated at the empirical rather than the theoretical level. On this basis, one research team has tentatively put forward the following 5-stage description of mathematical development for children in the age-range 11-16:

Stage 0. Children at this stage cannot use mathematics beyond direct application of small whole numbers, and geometrical transformations involving a single point.

Stage 1. Whole numbers can be viewed in pairs (coordinates), triples (vectors), as multipliers (ratio) or divisible (decimals). Children have begun to extend their mathematical vocabulary (eg tenths, halves, reflection) but are unable to apply this newly-acquired language.

Stage 2. The vocabulary acquired in Stage 1 can be used, and is further extended (eg decimals, vectors). Problems may require two steps for completion or need an interpretation in the form of a graph or the application of a formula. The child is not simply limited to carrying out an instruction, he is in a position to see what is needed for solution. Although not required to invent a problem-solving strategy, he has to ask himself "what method do I use"?

Stage 3. This shows the first emergence of abstraction and the formulation of problem-solving strategies. Decisions may be made about the size of an answer or of a measuring unit.

Stage 4. Mathematics appears as an abstract system and not merely a way of quantifying real world phenomena. A fund of knowledge is applied to the solution of problems: children can go beyond "child methods" and select from "taught methods" which have previously been assimilated and stored.

(These descriptions are adapted from Chapter 13 of K Hart (ed.), Children's understanding of mathematics: 11-16, pp 187-207. This text gives fuller details of the test questions used in the research, and children's responses to them. Copies of this book were available at the seminar.)

The results of their experiments suggest that, with existing textbook and teaching resources, the proportions of children at ages 13, 14 and 15 who successfully answered the test questions representing these various stages were:

	Age 13	Age 14	Age 15
Stage 1	over 70%	over 70%	over 80%
Stage 2	48-70%	50-70%	58-80%
Stage 3	18-48%	22-50%	30-58%
Stage 4	2-18%	5-22%	8-30%

These findings, which are generally in line with the results of a sequence of national surveys carried out by the Assessment of Performance Unit of the Department of Education and Science, clearly have important implications for curriculum design, including the question of minimum content. Particularly significant is the small amount of improvement recorded over the three successive year-groups, which suggests that although some broadening of the curriculum with increasing age may be practicable, for many children attempts to push the mathematics to a higher conceptual level are unlikely to be successful. The research team comments that:

"the vast majority of children even at age 15+ are not able to deal with abstractions and application in mathematics ... only about 10 per cent of our child population seem to appreciate even the most elementary concepts in the type of mathematics which could be described as a system of abstractions"

and concludes:

"It is impossible to present abstract mathematics to all types of children and expect them to get something out of it. It is much more likely that half the class

will ignore what is being said because the base on which the abstraction can be built does not exist. The mathematics must be matched to each individual and teaching a mixed ability class as an entity is therefore unprofitable. Indeed, in a class where the children are supposedly matched on attainment there will still be a wide range of individual needs." (Ibid, pp 203,204,210).

Put another way, what is being said is that, although a good curriculum does not necessarily generate successful learning, a bad curriculum will lead to almost certain failure. Experience in England is that one of the principal causes of this is the setting of over-ambitious targets. To quote again from the report of the Cockcroft Committee:

"researchers (into the mathematical needs of adult life and of employment) were told many times that too many topics were covered in mathematics, that the pace of teaching was too fast, that too little time was given to important aspects of the subject. We believe that if pupils follow a course whose content is better matched to their level of attainment and rate of learning, they will achieve not only greater confidence in their approach to mathematics but also greater mastery of the mathematics which they study. This should contribute to improvements in attainment, attitudes and confidence and so to a raising of standards overall." (p 145).

Besides the implications of this for the objectives of the course as a whole, it is also important for the phasing of mathematical development through the course. There are two areas which are of particular interest in this respect:

- (1) Topics which are required for other subjects, especially science, at a stage which may be judged by mathematics teachers to be premature for many pupils. Examples of these might include the division of numbers in decimal form, algebraic notation and trigonometry.
- (2) The need to define with special precision the curriculum targets at points of transition from one type of school to another (eg at the end of the English primary school). There is a tendency to try to round these off into a tidy form (eg "the four operations applied to whole numbers, fractions and decimals") which forces an excessive amount of new work into the final year of the "lower" of the two schools. It should perhaps be a general principle of curriculum design that the last year of each stage of education should be one of consolidation.

Final remark

The concentration in this paper on the needs of pupils at the lower levels of mathematical attainment should not be taken to imply that there is no interest in England in other aspects of curriculum planning, particularly those relating to outcomes (3) and (4). Strengthening the logical element in mathematics teaching, introducing more realistic applications of mathematics into the classroom, integrating calculators and computers into the mathematics curriculum, are amongst the other issues in which curriculum developers are currently involved. However, the questions discussed here are receiving considerable attention at the present time, and the English delegates welcome the opportunity to discuss them in depth with Soviet colleagues at the seminar.

APPENDIX

Summary of the Cockcroft Committee's foundation list of mathematical topics

The list is prefaced by a statement stressing the need for variety in the pupils' mathematical experience, and the importance of associating calculation with measurement.

The idea of positive and negative whole numbers, understanding of place value in writing numbers, ability to carry out simple calculations (using a calculator if appropriate) and to associate the correct operation with a practical problem.

Decimal notation, with the operations of addition and subtraction in practical situations.

The language and use of simple fractions, and their decimal equivalents.

Confidence in handling money, including the applications of arithmetic to simple money calculations.

The idea of percentages, and calculations with simple percentages.

Competence in using a simple calculator.

The use of simple formulae expressed in words.

Ability to tell the time and to do simple calculations involving time, use of a timetable, simple problems involving speed.

Measurement of length, weight and capacity with understanding of the appropriate units.

The idea of simple rates involving two measures.

Reading meters and dials.

Construction and use of tables and graphs, including the use of simple flow-charts.

Familiarity with basic spatial concepts in 2 and 3 dimensions, including scale drawings and plane representations. Perimeter, area and volume.

Understanding of simple mechanical movements.

Acquaintance with simple ideas of ratio and proportion in practical contexts.

Randomness and variability, and the idea of probability and average.

Mr Quadling highlighted his four aims A1-A4, and the research on understanding with the four inferred general stages. In response to a question about the implications of the demands of work, he said that such needs should be provided in the schools by a general foundation and any genuinely common skills, and more specialised needs should be provided at the work place itself.

An interesting suggestion was made that the table of stages presupposes a common programme of mathematics teaching. Is it assumed that it represents material which is the total of what should be taught at these ages? Or does it represent only an average of the successes and failures of various projects? Could other figures be obtained by looking at other projects? In reply, Mr Quadling said that the sample covered a variety of school courses and the tests covered material common to all. Regarding the implications of this research, it appears that only about 10 per cent of pupils are capable of getting much out of the presentation of abstract mathematics.

Finally, the Soviet participants suggested that the Cockcroft basic mathematics list essentially represents a "minimum" coverage, which had been a major topic of debate at the first Anglo-Soviet Seminar in Oxford in 1981.

PLANNING THE OUTCOMES OF MATHEMATICS EDUCATION

Mrs L V Kuznetsova, Institute for Scientific Research into Content and Methods of Instruction.

In recent years in the Soviet school there has been a transition to universal compulsory secondary education. This move has initiated a qualitatively new stage in the development of our schooling, characterised by a change in the social demands placed upon it by society. Previously, when the compulsory level of education was limited to 8 years, naturally enough the main task facing the secondary school was that of preparing its pupils for higher educational institutions. Now this task cannot be seen either as the only one or as fundamental. The transition to universal secondary education made it necessary to re-examine the purposes of schooling, both the general purposes and the particular ones relating to particular subjects.

The right to secondary education is guaranteed to every citizen in our country by its fundamental law - the Constitution of the USSR. This presupposes not only the opportunity to attend school and obtain documents testifying to the completion of the course, but also the actual achievement of a certain guaranteed level of learning. When it guarantees an education, the government has the right to expect a definite result.

This means that the educational community must face the task of planning this result with precision, covering all aspects of education - academic training, up-bringing and development. In other words, some model of general education must be created which specifically established the level of learning, upbringing and development which the school is obliged to provide for each of its pupils.

The present paper is concerned with the educational aspect of this model. When we speak about the results of mathematics teaching we shall have in mind the actual mathematical training of school pupils. Returning to what has been said above, we would emphasise that the planning of the outcome of teaching must be seen as a means of realizing the rights laid down by the constitution of the USSR.

The term "planned outcomes of education" was introduced in our country by I I Kulibaba. He proposed that the concept should cover the knowledge, abilities and skills required of pupils and be presented in the form of a description of the intended results of education. An essential feature of his work was his proposal that the demands to be met by pupils should be included in the syllabus.

These proposals were translated into reality a few years ago when all the laboratories of the Institute for Research into the Content and Methods of Instruction of the Academy of Pedagogical Sciences of the USSR drew up draft syllabuses. In the mathematics syllabus it did not prove possible at that stage to present in full the whole systems of aims of teaching this subject.

Attempts to work out in practice the requirements of mathematical training at various levels (the general educational level, the level of preparation for higher education and others) were made by various researchers at various times. This research, while it provides some useful examples of the solution of the problems of defining requirements for knowledge and skills, does not solve the general problem of planning the outcome of education on the general school level, because it is fragmentary and based on a variety of approaches to the solution of the problem (which differ from each other both in conception and content). At the present time this important problem is being explored in the mathematics teaching laboratories of the Institute.

Planning the outcomes of education, like any scientifically based planning, includes three aspects: working out the planned outcome of teaching, organising pupils' achievement of these results, and monitoring. The first aspect is the most fundamental. On the nature of its solution depends the possibility of carrying out the other two. Starting from this point, we take the following line.

The planned outcomes of mathematics education must be a component in the syllabus (or, more broadly, in the set of syllabuses) for this subject. The earlier proposal that it would be expedient to include in the programme what is required of pupils in the way of knowledge and skills at this stage takes on particular relevance. It is only possible to guarantee a particular level of training for a pupil when that level is prescribed by law. And the weightiest legal statute for schools is the set of syllabuses.

Another important reason why it is essential to include requirements in the syllabus is that they can have a strong organising influence on the whole learning process. Requirements which are set out in the syllabus (as against those which are formed incidentally) make it possible to regulate this influence and to make it more effective.

The necessity of providing all pupils with a mathematical education is not dependent upon the directions they will follow on leaving school, and if a certain guaranteed level of training is to be reached the first task must be to define minimal planned results, the achievement of which is compulsory for all pupils.

The planned outcomes of education are set out in the syllabus in several stages by means of the gradual concretisation of the more general levels in the description of the aims of teaching. There are three such levels mentioned in the syllabus. On the first of these the educational tasks for the corresponding course are formulated in a very general form (mathematics IV-V, algebra VI-VIII, geometry VI-VIII etc). The basic direction of the course is defined, the main course content is concisely stated, and the position of the course in relation to the other mathematical subjects is clarified.

On the second level, the general requirements which mathematical training must meet are set out. These requirements, which closely reflect the general aims of the course, are spread across the content of the syllabus (calculation, equations and inequalities, measuring and constructions etc). For a description of the requirements in the syllabus a work-list approach is adopted which is dependent upon the particular features of the discipline in question. It reflects the traditional method of mathematics teaching in which pupils master a whole series of mathematical skills. Recently this approach, it is true, has been undergoing certain changes. Greater emphasis than before is beginning to be placed on students' theoretical knowledge and for formation of mathematical concepts. However in the last analysis these aspects of mathematics education express themselves in the ability to apply the theoretical knowledge which is being built up; as well as serving as a means of developing pupils' mathematical thinking and broadening their mathematical outlook, they also serve as a means of improving the corresponding skills. Therefore it is appropriate that the requirements are described in terms of skills which pupils must master by the end of each course.

At this level what is required of pupils is still indicated in a fairly general form, and various interpretations are possible as regards the degree of mastery of the skills in question. The main purpose here is properly to direct the work of teacher and pupil and the authors of textbooks, methodological handbooks and other teaching means. Links within the subject and between subjects are indicated and applications are characterised.

The next stage in the concretisation of the requirements of mathematical training is reached through the definition of the level of completeness which must be reached for each of the skills identified at the previous stage - the actual planned outcome of mathematics education. The main question here concerns the form in which this level should be set out. Obviously it must be set out constructively so as to make it possible to check the extent to which it has been achieved. The most obvious solution is to set out the planned outcome of education in the form of a list of problems typical of the required level of mastery of the knowledge and skills in question.

This way of setting out the planned outcomes of education is a perfectly natural one. In the mathematics course problems have traditionally been seen as the ends of the study of the subject and have always served as a way of checking pupils' grasp of the material. Moreover, recently in methodics the view has become established that problems also serve as a basic means of mastering theoretical material. Therefore to set out the planned results of teaching in the form of a list of problems adequately reflects both the learning process whereby knowledge is acquired and the ways of checking the extent to which these results have been achieved.

Like the requirements of mathematical training, the planned outcomes of education are spread across the content of the syllabus and describe the end result for each stage of teaching. In that they set out the actual level of mastery of mathematical knowledge and skills they constitute a direct concretisation of the general requirements of mathematical training. A necessary condition is that they should reflect each of the requirements formulated at the previous stage.

There are a number of reasons why the syllabus is oriented towards the end results of education. The process of the acquisition of knowledge and skills is one which is extended over time. Corresponding knowledge and skills are formed and consolidated over a fairly long period. Therefore the basic criterion for assessing the results of teaching is naturally found in the state of knowledge at the end of a certain stage of teaching and not in the intervening phases.

A description of the end results of mastering the syllabus material makes it possible to provide continuity between various stages of learning and also between parts of the syllabus with related content.

End requirements may be formulated independently of the methodological approaches adopted in various school textbooks. Sometimes the introduction into schools of various types of regulations may be seen as limiting the teacher's initiative in the learning process. But if the planned level of learning which must be reached at the end of any stage of teaching is set out in the syllabus, this leaves the teacher completely free to choose the means and methods of achieving the required result and in no way restricts the authors of textbooks and methodological handbooks in the way they choose to provide material to the required level.

Setting out the planned outcomes of education in a very concrete form with the help of a list of problems makes it possible at this stage to identify the various levels of mastery of

the material, amongst them, first and foremost, the compulsory level, the achievement of which is essential for every pupil and guarantees a good mark. Here there is an important difference between the proposed approach to the planning of outcomes and previous attempts to state the level of difficulty of concrete tasks. The latest approach requires that a level of mastery of the material should be achieved which is not lower than the proposed level, whereas previous models were directed at a level not higher than the proposed one. This led to variations in the interpretations of the requirements, the absence of a precise lower boundary which would guarantee pupils a good mark, and hence disparities in the assessment of pupils' knowledge and skills.

The necessity of providing all pupils with school mathematics education presents the school with the task not only of providing a certain minimal level of training but also of making a fuller and more profound mastery of the material possible so that pupils may continue their mathematical studies after they have left secondary school. Therefore along with the minimal planned outcome the syllabus must establish a more advanced level of mastery of the material.

At the present time in the mathematics teaching laboratories work has already been carried out on the formulation of the general aims of every course, and the general requirements of mathematics teaching which are formulated in the secondary school syllabus have been worked out. Now work is being done on the definition of the minimal outcomes of mathematics teaching, the achievement of which is compulsory from the point of view of the general educational aims of the course and provides real possibilities both for the further study of mathematics and also for the use of the acquired knowledge in the study of related disciplines. This work has already been completed in relation to individual topics in the school mathematics course. In the future we intend to carry out similar work for the entire school course.

Mrs Kuznetsova said that her paper reflects the work of a whole group of people in the Mathematics Teaching Laboratory at the Institute. They are concerned with the difficult task of translating the general aims of the mathematics curriculum into a concrete form related to each particular stage of the school programme.

The issue of achieving predetermined levels in mathematics education is bound up with the definition of the aims of school mathematics, which in their turn reflect what society demands of its schools. The general aims have to be transformed by a multi-stage process into specific learning/teaching tasks, and even into specific mathematical problems.

However, although this is the most direct method, it has its pit-falls and can lead to a loss of important aspects. The process can be reversed, to study how particular activities realise specific aims, and these in turn reflect more general aims. Such a trial and error process is more difficult, though it may in the long run be more reliable than the 'top-down' pathway. (Editor's note: The Seminar was characterised by the use of a considerable number of English proverbs, the appropriate one here being 'the longest way round is the shortest way home'.)

The English concern with differing rates of advancement of pupils, which usually implies that different pupils proceed at different speeds through the syllabus in England, is taken into account in the Soviet Union by having different levels of mastery of a particular skill.

The British participants found Mrs Kuznetsova's paper most valuable. They suggested that one can only evaluate the outcomes of school mathematics education in a socially valid way by studying what happens to pupils over a prolonged period after they have left school. In this way the theme of this paper relates very closely to the research described in the paper by Mr Fitzgerald.

Session 7. The New Geometry Course in USSR

AFTERWORD ON THE METHODOLOGICAL POSSIBILITIES ARISING OUT OF
A V POGORELOV'S TEXTBOOK

I F Teslenko and V V Firsov

Over ten years ago Academician A V Pogorelov wrote his now famous book 'Elementary Geometry' which within a very short time went into several editions. The book was written as one version of the geometry textbook designed for use in the majority of schools. As a strong supporter of the traditional system of presentation which drew upon centuries of experience of geometry teaching, A V Pogorelov chose as a model the popular geometry textbook produced by the Soviet pedagogue A P Kiselev. Of course, from the scientific point of view this textbook was by that time already out of date: first and foremost it was necessary to re-examine the system of proofs so that they were fuller and better substantiated and to include a whole range of new sections which would widen the range of ideas in the course (geometric transformations, coordinates, vectors and so on). However, from the methodological point of view A P Kiselev's textbook was well designed: the order of exposition of geometric material chosen by him was properly structured and facilitated effective use of the wealth of practice material. The methodics of teaching the proofs of geometric statements, based on the use of the conditions for the similarity of triangles, since it was extremely clear and therefore accessible to pupils, was of particular value.

Now we can only guess at how much simpler it would be for schools to make the transition from the structurally traditional textbooks of the pre-reform years to A V Pogorelov's textbook: unfortunately at that time the general enthusiasm for constructing courses on the basis of geometric transformations did not permit of the possibility of even examining Pogorelov's version. In time, however, two fundamental faults in the new approach became clear: the difficulty which pupils in class 6 encountered when they tried to grasp the basic object of their study - the geometric transformation, and the fact that the corresponding system of exercises had not been properly worked out.

The serious criticism of the state of mathematics teaching in schools which has been heard in the last few years in the statements of prominent scholars, in numerous letters from teachers and parents and in articles in journals and newspapers, bears a direct relation to the teaching of geometry. It is an accepted fact that the existing geometry textbooks are difficult for pupils and teachers alike. Although much time was devoted to their preparation, the books have not yet led to any real rise in the level of pupils'

geometric training. All of this has made it necessary to look with new eyes at the values of the traditional system of presentation as realised with the now essential scientific rigour and fullness in A V Pogorelov's book. Therefore, in accordance with the proposal of the Committee for the Reform of Mathematical Education of the Mathematics Department of the USSR Academy of Sciences and the Ministry of Education of the USSR, A V Pogorelov prepared a new edition of his book reworked for schools and intended for experimental use. The results of the experiment which was carried out in many schools in the Ukraine using the new edition turned out to be very promising. Within a very short time A V Pogorelov prepared a further edition of the textbook which took into account the remarks and suggestions of teachers and methodologists; this was also subjected to trials. And now we have before us a third version of the textbook. The aim of this edition is to acquaint a broad spectrum of the teaching community with a prospective version of the geometry textbook being prepared for widespread use in schools. A resolution of the Board of the Ministry of Education of the USSR recognises A V Pogorelov's textbook 'Geometry' for classes 6 to 10 as suitable for use throughout the Soviet Union and recommends that it should be introduced on a wide scale in schools beginning with the academic year 1982-83 (an edition of 1,150,000).

What sort of textbook is this? What are its mathematical and pedagogical features? How is it to be used? These are the main questions which concern teachers. It must be said that as regards many of its parameters the textbook is unorthodox and idiosyncratic and the methodological views of the author are difficult to identify from the text and require explanation and substantiation. The purpose of this afterword is precisely this: to acquaint the reader with our understanding of A V Pogorelov's methodological system, an understanding based on numerous conversations and discussions with the author and on an analysis of the progress and results of the experiments in the use of the textbook carried out over many stages and on a large scale in schools in Kharkov and the Kharkov region, Sebastopol, Kiev, and in the Moscow region.

A V Pogorelov's professed aim in teaching geometry can best be described by a quotation from his foreword to 'Elementary Geometry': 'In offering the present course we have taken as our starting point the idea that the main purpose of teaching geometry in school is to teach the pupil to reason logically, argue his statements and prove his point. Very few school-leavers will be mathematicians, even fewer will be geometers. There are also those who, in their practical activity, will never use Pythagoras' theorem. However, it is highly improbable that there will be any for whom it will not be necessary to reason, analyse and produce proofs.'

On the basis of this view A V Pogorelov's textbook represents a systematic presentation of the geometrical content required by the syllabus - both traditional topics of plane and solid geometry and sections which are new for schools - built upon an original system of axioms.

There is a well-known view according to which pupils in the 6th class are quite incapable of grasping the axiomatic structure of geometry. As J Dieudonné wrote: 'I believe that no axiomatic system whatsoever should be introduced into school before children have reached the age of fifteen'. The conclusions of psychologists are sometimes adduced in support of this view; however, these conclusions are not uniform and they can be interpreted in different ways. Of greater importance are our notions about geometrical axiomatics in general. It is obviously pointless to attempt to construct a geometry on the basis, for example, of Hilbert's system of axioms. More economic systems of axioms avoid the difficult question of the introduction of measure for lines and angles by means of an axiomatic introduction of measure.

Thus you may be sure that the system of axioms chosen by A V Pogorelov is very economical. The axioms for the measurement of lines and angles make it possible to relate many of the difficult areas to the arithmetic of real numbers with which pupils are already acquainted.

Opponents of the axiomatic structure of school geometry often confuse two points -the actual structuring of geometry on the basis of a particular system of axioms and the pupil's recognition of the idea of such a structure. Should we require a pupil of class 6 to grasp the axiomatic construction in his very first lesson? Of course not. In full accordance with this thesis the opening of A V Pogorelov's textbook (to be more precise the introduction to the systematic geometry course) does not contain a single axiom! Here the fundamental properties of the simplest geometrical figures, with which all pupils are well acquainted from experience and from their previous study of mathematics, are systematised, and simple reasoning processes are followed through in the course of which other properties are deduced from the original ones. Subsequently these fundamental properties will be declared axioms, but neither the teacher nor the pupil is aware of this! Thus an understanding of the logical construction of geometry develops in pupils by gradual stages; it is formed in the course of learning.

The suggestion is often made that the period of introduction of systems of axioms should be considerably extended: at first all work should be on the level of visual representations and intuitively understandable conclusions; only when the first real results have been

achieved in this way should the transition be made to the precise formulation of an axiomatic exposition. But if this is done too late the fundamental course content will be presented on a level which is so basic as to be no longer suitable. Nevertheless there is a rational core to the suggestion we have been discussing, namely that some propaedeutic work must precede the systematic presentation of geometry.

We would emphasise one important advantage of a sufficiently early introduction of axiomatics. It is connected with a pedagogical question of principle: the reason for the necessity of proofs. It is one thing when we take it as necessary to prove everything and the authors of the textbooks themselves follow this procedure. Having thus stated 'rules of the game', we then have, albeit not in an obvious form, the idea of the logical order of geometric facts - the pupil can fall in with this. It is quite another thing when some statements are proved while others like them are not proved, either on the grounds that they are self-evident or because it is too difficult to prove them.

Let us consider the question of the proof of self-evident statements. Within the framework of the first approach - the adoption of the procedure of proving everything without exception - the proof even of self-evident facts seems to the pupil to be justified. On the other hand permission not to prove the obvious may destroy the justification of the necessity for proof in general: for if it is possible not to prove one thing on the grounds that it is self-evident then it is possible to take other facts on trust as well. The adoption of the 'prove everything' approach creates in pupils a powerful psychological sense of purpose which is particularly important in the early stages of learning. Therefore it is particularly important in the early stages of systematic exposition scrupulously to follow logical links. Subsequently when pupils have already found their feet, it is perfectly permissible to lower somewhat the requirement for formal completeness of proofs (assuming that in every case it is possible to carry out the complete proof).

In A V Pogorelov's textbook the 'programme' for carrying out proofs appears at the end of the introductory paragraphs - after the repetition of the fundamental properties of the geometric figures and at the same time as the introduction of the term 'axiom' and the conferring of "axiom status" by means of the fundamental properties which have been singled out. The traditional structure of the course, in which the basic apparatus of proof is the use of conditions for the equality of triangles, means that proofs argued directly from axioms are not carried out for very long: a fairly rapid transition to theorems about the equality of triangles puts into the hands of pupils an effective apparatus of proof. After the appearance of the conditions for the equality of triangles the axioms themselves are only very rarely incorporated in proofs. But it is at this stage that we begin to reap

the fruits of the timely carrying out of the 'programme' of proofs which has prepared pupils for the necessity of producing proofs, and once they have at their disposal the apparatus for proof - conditions for equality - they are already prepared for the solution of a number of interesting problems.

The beginning of the systematic geometry course has always caused pupils difficulty. They face a particular psychological barrier which must be surmounted. Just as when young children learn to speak they make mistakes and construct improper sentences, so in the experiment on A V Pogorelov's textbook pupils in the 6th class learned to produce proofs. At this stage of learning their right to make mistakes must be recognised (though of course the mistakes must be corrected by the teacher); they must be helped to look constantly beyond temporary difficulties towards the fundamental goal - that of fulfilling the programme to prove everything. The moment of reward arrives: in all the experimental classes after the appearance of the conditions for the equality of triangles the children began to talk! Of course a long road still lay ahead in the teaching of proof, there were mistakes, but the idea itself had been grasped and the desire to follow it through was apparent.

The system of axioms selected by A V Pogorelov makes it possible to construct the whole basic core of the content of the geometry course - the totality of geometric facts - in a strictly deductive manner. Proof reaches a high level. Thus for example many properties of motion (displacement) which in the existing textbook are accepted without proof are here proved.

Nevertheless the course is not cumbersome. The apposite system of axioms, the logical order of presentation and also the new mathematical approaches which the author has found for presenting a number of difficult sections, have made it possible to keep down the size of the textbook. The effective linking of geometric statements arising from the coordinate method to proof significantly decreases the volume of material, is absolutely modern in spirit and creates the preconditions for the making of inter-subject connections between geometry and algebra. The reader can see for himself the extent to which the use of the arithmetic approach in the presentation of the subject of vectors simplifies the proofs of many traditionally difficult questions such as, for example, the distributive property of the scalar product. In A V Pogorelov's exposition the proof of this property becomes a simple exercise even for the weaker pupils!

As we have already noted, the core content of the textbook is organised in a strictly deductive manner but on the periphery of the core content there are deviations from this rule arising out of methodological considerations. Thus these considerations result in the

subject of geometrical constructions by means of ruler and compasses being dealt with in the 6th class. At this stage it is not yet possible to prove for example that if the distance of a straight line from the centre of a circle is smaller than the radius of the circle then there are exactly two points at which they intersect. But for real constructions and their bases this is not necessary. Therefore when the subject of geometric constructions using ruler and compasses is presented details of this sort are omitted. Similarly many nuances in the subject of measurement of geometric magnitude (length, area and volume) are ignored. The strict exposition of the subject of measurement of magnitude is full of subtle nuances and requires refined apparatus which is beyond the scope of the school syllabus. It is completely natural and acceptable to lower the level of the logical requirement for this applied subject.

Many purely mathematical details of the presentation will be familiar to the teacher who has taught and has been taught using the old textbooks. Others may initially give rise to misunderstanding. Thus for example (as it is not the case in the textbook being used at present) the triangle is defined, not as part of a plane, but as a figure formed by three points which do not lie in a straight line and three lines connecting each point to another point. If we think about it those are exactly the sort of triangles we draw on the blackboard or in an exercise book.

Another example is the definition of equality of triangles. In the textbook in use at present the equality (congruency) of triangles is defined by means of displacement; in the old geometry textbooks the equality of triangles was defined by means of superposition. The first of these definitions is too abstract for pupils in the 6th class; the second is difficult to use in the axiomatic exposition because it requires the adduction of correspondingly complex axioms.

In A V Pogorelov's textbook an interesting solution to this problem has been found: the equality of designated triangles is defined. In other words according to the definition $\triangle ABC = \triangle MNP$, if their corresponding elements are equal: $AB=MN$, $AC=MP$, $BC=NP$, $\angle A = \angle M$, $\angle B = \angle N$, $\angle C = \angle P$. Here the order in which the elements are written down is significant: ' $\triangle ABC$ may not be equal to $\triangle BCA$ '.

The adoption of a definition of this kind immediately excludes a number of purely mathematical difficulties and simplifies the proof of many statements. Any discomfort experienced by the teacher (but not by the pupils) quickly passes. On the contrary, the well known methodological advantages of this kind of approach are revealed: from the very outset pupils learn to take note of which elements of the triangle are equal to each

other. Subsequently, after the introduction of geometric transformations, the concept of the equality of triangles is generalised: figures are said to be equal if they can be translated one onto the other by movement.

The final definition of the equality of figures coincides with the definition adopted in the textbooks used at present. However, it is presented to the pupils at a considerably later stage, when they have already built up a large store of knowledge about geometric figures and their properties.

We would also draw the attention of the teacher to the traditional nature of the symbols used and to the absence of demarcation between a line and its length, an angle and the measurement of an angle etc. - demarcations which are always obvious from the context in which they are written. Practical experience shows that the deliberate lowering of the formal requirement in relation to this subject does not lead to confusion but makes learning considerably easier by ridding it of non-essential details. Moreover the absence of the said demarcation encourages pupils not simply to write things down formally but to analyse the content of what they write - Does it relate to the angles or their magnitude? for example. We rid ourselves of the need to waste time and effort trying to get pupils to understand and master complicated symbols and we avoid a huge number of the mistakes which they would otherwise make.

The final feature which should be mentioned in this discussion of the mathematical aspects of the textbook is the widespread use of constructive definitions. The constructive definition of a mathematical object, like the definition of a triangle given above, enables the pupil to visualise the process of construction of the object. This makes the formation of concepts much easier. It is true that definitions via genus and the distinctions between species look shorter but if they are to be grasped it is necessary to carry out a complex mental operation involving the identification of a certain sub-set of objects (species distinction) from the description of the given set (genus). It is well known that the formation of concepts which are defined in accordance with this schema requires special methodological work and takes up a great deal of time.

In this brief survey we are not able to deal with all of the original methodological features of A V Pogorelov's textbook. This is all the more true because the author's pedagogical ideas concerning the organisation of the pupil's academic activity are, in our view, of greater interest. In any case without an acquaintance with these ideas it is impossible to gain a proper impression of the textbook, to judge it as it should be judged and use it to the desired effect.

The reader will already have noted the surprisingly small size of the book. The textbook on plane and solid geometry for the 5th class of secondary school, which contains all the necessary exercises and revision questions, covers only 261 pages of a standard-sized volume. Of course, the mathematical inventiveness of the author is one reason for the diminished size of the textbook, but it is not the only reason.

A V Pogorelov's textbook embodies a new pedagogical conception of a textbook which has been advanced only in very recent times by a number of Soviet pedagogues and therefore it is difficult to judge it if one sets out from traditional ideas about textbooks.

Using pedagogical jargon we may say that the traditional textbook is 'methodicised'. In it everything is written just as it should be done in the lesson: it is necessary to read a section of text aloud, then do the relevant exercises, then read the next section of text and so on.

The textbook as it were becomes the chief organiser of the pupil's activity, while the teacher reproduces the text of the textbook and checks that it has been mastered. Of course, we have consciously exaggerated the picture somewhat in order to throw A V Pogorelov's approach into greater relief. Moreover, as far as we know, not one of the mathematics textbooks has ever succeeded in realising the idea of 'methodicisation' to anything like a sufficient extent.

Supporters of this conception believe that the best textbook is the one in which the pupil can study a large part of the material completely independently. Hence the ideal would be to work completely without a teacher, and the best textbook would be of the teach-yourself variety. This idea seems to us pedagogically unfounded, and we believe that attempts to put it into practice to any extent would be dangerous for the school.

The teacher has always been, and still is, the central figure in the pedagogical process: he is the main organiser and guide in the learning process, the main source of academic information. Only the teacher, who spends his time in the classroom watching and assessing the reactions of his pupils, can pick the optimal speed and level at which to teach.

Each new lesson on the same subject is created anew by the teacher, and here we see revealed both the creative nature of the process of educating and bringing up pupils and the creative nature of the teacher's work. Is it therefore justified, given such an approach, to assign to the teacher the role of reader and reproducer of the textbook?

The above mentioned idea did have some positive significance when schools did not have a sufficient number of qualified pedagogues. Now the situation has changed: increasing numbers of people are seeking work as teachers and it is not acceptable to relegate teachers to the humble position of textbook commentators.

The central role of the teacher is strongly emphasised in the idea of the textbook which A V Pogorelov advances. According to the author, the textbook is a book to be used by pupils for independent work after they have listened to the corresponding explanations by the teacher. In other words, it is a book for the revision and consolidation of that which has been covered, to be used independently by pupils. Therefore the textbook lacks the traditional methodological material - all sorts of explanations, introductions, reasons for introducing concepts, preparations for the proofs of theorems, and a great deal more that although an essential part of a lesson is superfluous given a textbook of this type.

One might ask what possible harm there could be in including this material in a book for consolidation and revision? But practical experience in education indicates that too much text makes it difficult for pupils to isolate what is really important in the text. And so, during the lesson the pupil makes notes of the proof and later studies not from the textbook but from his notes. This conception of the textbook requires completely new methodological provision. The teacher's book becomes much more important, for in it must be contained all the methodological material from the textbook, various versions of the exposition, methodological recommendations for the organisation of study and so on. In other words, the teacher's book must be an expanded textbook which gives a variety of versions of the exposition and recommends the use of various teaching methods.

We shall show by one very important example how this approach is applied. Let us examine the strict proof of the theorem of the sum of the angles of a triangle (p 37 of the textbook).

In this proof, all the necessary arguments are adduced for the sake of logical strictness and completeness. Consequently the proof looks cumbersome and is hardly likely to seem comprehensible to pupils: the logical nuances of the proof's base will conceal from them the simple and obvious meaning of the geometric picture. But this is the final proof, with the reproduction of which the study of the theorem is supposed to end.

The following procedure is recommended for the lesson. First present pupils with an abbreviated visual proof, omitting some of the logical arguments when their meaning is clear from the drawing and visual understanding: this is the first stage in the study of the

theorem. When the idea of the proof is understood by the pupils, the proof is repeated, with the addition of and particular attention to the arguments which were omitted - this is the second stage in the study of the proof. Finally, in the third stage the proof is reproduced in full, in the form in which it is cited in the textbook.

We shall illustrate this methodological device with the proof of the theorem of the sum of the angles of a triangle.

First stage Let ABC be a given triangle. The mid-point O is marked on the side BC of the triangle. Line AO is produced to D , so that OD is equal to AO . Now the triangles BOD and COA are congruent: the angles at O are equal because they are vertically opposite (*) and $OB = OC$ and $OA = OD$ by construction.

From the congruency of triangles BOD and COA it follows that the angles DBO and ACO are equal. But these angles are alternate for the straight lines AC , BD and the transversal BC (*). This means that the lines AC and BD are parallel.

Since the angle ACO is equal to the angle DBO , the sum of the angles of our triangles at vertices B and C is equal to angle DBA (*). This means that the sum of all three angles of the triangle ABC is equal to the sum of the angles DBA and CAB . But these angles are internal angles between parallel lines AC and BD and are on the same side of the transversal AB (*). Therefore their sum is 180° and therefore the sum of the angles of triangle ABC is 180° .

In the second stage the proof is repeated and in the places marked by asterisks questions are put and the necessary amplification is provided. The traditional school proof of the sum of the angles of a triangle (in one of its versions) differs from the one used in the textbook only in that the arguments marked with asterisks are not adduced -hence its lack of strictness.

The teacher's book contains recommendations for carrying out each of the stages of a proof. In the textbook, which the pupil will use at home after he has heard the proof three times in the lesson, there is, of course, only the final version of it.

From this important example we see that the content of the lesson is not just a copy of the content of the textbook: it is richer and more varied from the methodological point of view.

Experiment has shown that although at first the teacher is unaccustomed to this sort of approach he soon masters it and begins to become aware of its advantages over the traditional one. We note in particular the psychological benefit, identified by many of the teachers in the experimental classes, which this approach provides for the teacher: it is not particularly pleasant only to tell pupils the things which they could read for themselves in the textbook without the teacher's help.

However it is not enough just to tell pupils that the textbook is intended for their independent work. It is also necessary to make provision for the organisation of that work.

In A V Pogorelov's textbook a very original system of revision questions serves this purpose. Of course revision questions are contained in many textbooks. However their role in A V Pogorelov's book is unique. The pupil may be sure of the following: a) for any question in the textbook there is a precise answer - it is just a matter of finding it; b) all the questions taken together cover the entire content of the textbook. Thus the system of questions picks out the most important of the theoretical material studied while at the same time fixing the level at which it must be mastered. Homework on the theoretical material can be set not in the traditional manner of "learn that section" but in a concrete way: "prepare answers to questions 12, 13 and 15".

This system is very convenient for teachers and pupils alike. The precise nature of the task is important, as is the fact that the task is a finite one. The pupil knows what work he has to do and how to obtain a good mark - a powerful psychological factor which encourages the pupil to carry out the work. The teacher knows that if a pupil has not prepared an answer to the question set from the textbook this cannot be explained by the volume of work or an inability to pick out the important points from the text etc. And when pupils begin to work questions, they learn to work independently with the text, to see in it what is essential, to distinguish its elements - so in an unobtrusive way pupils are being taught to work from books. This, incidentally, helps any pupil who has missed classes to catch up on preparation independently. Of course in particularly difficult cases the teacher's help is required - but where is the textbook with which that would not be the case?!

The textbook also contains sufficient problem material. An important feature of it is the frequent "recycling" of the basic material in the solution of many problems. One and the same basic geometrical configuration, one and the same method of proof reappear over and over again in the proofs of theorems and the solution of problems, is repeatedly shown to pupils in the course of one lesson and consequently becomes fixed in their memory.

In general, in the early stages of teaching it is recommended that the teacher should resort more frequently to showing models for the pupils to follow without yet encouraging pupils to prove a theorem for the pupils to follow without yet encouraging pupils to prove a theorem or solve a problem independently. On the contrary, it is useful to show in class the solution of one problem and then ask that another analogous problem should be solved independently. Indeed, many of the problems in the textbook are in pairs so that this can be done.

There is still a great deal to be said about the pedagogical features of the textbook. Not all of what has been said above is true of the earlier editions of the textbook. Much was gained from experiment that revealed particular drawbacks in one version which could be corrected in subsequent versions. Indeed, there are probably drawbacks to this, the third, edition.

However even in its present form it is a very promising version of the secondary school textbook, attractive by virtue of its wide-ranging mathematical and pedagogical worth. We may conclude that an acquaintance with A V Pogorelov's textbook would be useful to every mathematics teacher.

Academician A. V Pogorelov himself attended half a day of the seminar in order to speak about his new geometry textbook, which is about to be introduced throughout the Soviet school system for Classes 6-10. He explained that it has three main aims:

1. to develop logical thinking in the student and skill in problem-solving;
2. to promote active mastery of the subject by the student;
3. to prepare and train the student for further study of mathematics.

To achieve these aims, Academician Pogorelov had adopted the following approach:

1. Geometry is a deductive science. Ten axioms are developed in the opening 30 pages of the text, drawing on the students' experience. Then the accent falls on deduction and proof, problem-solving, etc, with the axioms in the background.
2. Most of the time in the 350 lessons would be spent on problem-solving, and hence aim 2 is met.

3. Although it is customary for the university teacher to tell students to forget all that they learned at school, because it is incomplete, that cannot be so with this new book since the set of axioms is complete and the logical foundation secure.

In the discussion which followed, the main points covered included:

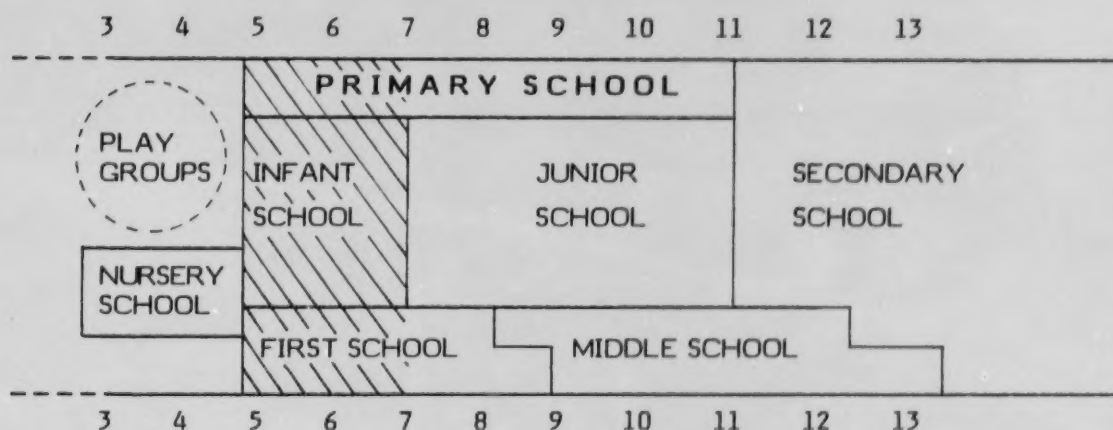
- a. Although there is work on geometry, including analytical and projective geometry, in the university course for intending teachers, there is less in the mathematics degree courses (except in optional courses), so the British participants had doubts about whether geometry of the kind in the book gave a good preparation for further education (aim 3). For example, would not more emphasis on spatial understanding be better for future engineers, etc?
- b. The book is not only an exposition of an axiomatic geometry, however. It includes a wide range of approaches, including some work on transformations and on 3D geometry. Trigonometry is used a great deal. Vector geometry of the plane is included, as is some integration.
- c. The completeness of the axioms is stated, but perhaps only in the Teachers' Guide.
- d. The author has also written a similar, but different, book for student teachers; this contains more background.
- e. When asked whether there was any evidence that students who had followed the book during its trial use did reason more logically in other fields, Academician Pogorelov replied that he had only subjective evidence, but there was no doubt that students were challenged by their fellows to substantiate statements and were more prepared to argue their case.
- f. It would be interesting to set up a research project to see if using this book did lead to more logical reasoning in other subjects, such as physics and geography.

Session 8. Some Specific Issues in Mathematics Education

MATHEMATICS IN THE INFANT SCHOOL (ages 5-7)

E A Albany, Director, Nuffield Mathematics 5-11 Project

In England and Wales the majority of children attend either a separate Infant School or the Infant Department within a 5-11 Primary School. The remainder of 5 to 7 year olds, who live in an area where the three-tier system consisting of first, middle and secondary schools is employed, are taught in the first two years of a First School. In general, the "Infant Stage" is taken to mean the age-range from 5 to 7 years.



In the British educational system the curriculum is seen as a tool in the hands of the teacher; this is in contrast to other educational systems where the curriculum is imposed upon the teacher. To a large extent the headteacher of a British primary school is autonomous in curricular matters. This means that within each school collective decisions have to be made concerning the selection and planning of an appropriate syllabus for each subject.

The mathematics programme may be designed by the teachers of the school acting under the leadership of the headteacher with advice from the local authority adviser. In some cases local authorities set up a committee of teachers and advisers to produce guidelines in primary mathematics which schools may choose to adopt totally or in part. These guidelines often refer to published texts or programmes, thus encouraging discussion among teachers on the relative strengths and weaknesses of these published materials.

The local teachers' centre arranges discussions, displays and demonstrations of various published programmes so that teachers are given the opportunity to inspect a scheme closely and try it out in their own school before making a decision to adopt it.

The factors which may influence a headteacher and his staff in adopting a published scheme are many and varied:

- (i) Is the scheme based on sound pedagogic principles and has it been tried out in schools?
- (ii) Is the pupils' text clear, at a suitable level of readability and closely linked to language development?
- (iii) Is the format of pupils' material attractive or are pages overcrowded and "fussy"?
- (iv) Is pupils' material expendable (for example workbooks or worksheets) or non-expendable (workcards)?
- (v) Is the teachers' handbook supportive, giving clear indication of stages of development, practical suggestions for teaching, home-made apparatus, record-keeping, analysing difficulties, etc?
- (vi) Is the scheme flexible enough to adapt to individual needs and class organisation or is it too prescriptive and rigid?
- (vii) Is there a balance between number, measurement and spatial work or too much emphasis on one at the expense of the others?
- (viii) Is enough practice and reinforcement material provided?
- (ix) Is the cost prohibitive? (Books and materials are provided free to all children; they are purchased by the local authority from an equipment allowance based on the number of pupils in the school).
- (x) Does the scheme lead on naturally to the work being done in the next school or department?

Several official publications offer guidance and comment on current trends in primary mathematics curriculum development. Here are a few examples:

"Mathematics Counts"

Report of the Committee of Inquiry into the Teaching of Mathematics in schools under the Chairmanship of Dr W H Cockcroft. (Published by Her Majesty's Stationery Office, 1982).

"It must be for each school to develop its own aims and objectives for teaching mathematics in the light of its approach to the curriculum as a whole; it must be for each class teacher to seek to achieve these aims and objectives by the provision of suitable activities for the children in the class".

"At the primary stage new topics and concepts should always be introduced by appropriate oral and practical work and the necessary links with what has gone before established by discussion".

"Language plays an essential part in the formulation and expression of mathematical ideas".

"The primary years are a time when children are not only acquiring the skills of language and number but are also experiencing a variety of methods of learning; they are learning to think, to feel and to do, to explore and to discover".

"The policy of trying to avoid reading difficulties by preparing workcards in which the language is minimized or avoided altogether should not be adopted. Instead the necessary language skills should be developed through discussion and explanation".

"Mental and oral work should play a major part in early learning and young children should not move too quickly to written sums".

"The ability to learn mathematics from the printed page is one that develops very slowly. In the primary stage new topics should always be introduced by appropriate oral and practical work".

Re practical work - "It is necessary to realise at the outset that such work requires a considerable amount of time. However, provided that the practical work is properly structured with a wide variety of experience and clear stages of progression and is followed up by the teacher by means of questions and discussion, the time is well spent".

"The measurement of length, capacity, weight, area and time should be part of every primary child's experience".

"Mathematics 5-11: A Handbook of Suggestions. (D.E.S)

(Published by H.M.S.O., 1979)

"The role of language in the learning of mathematics is clearly recognised in many mathematics schemes in use in schools today. Current schemes (as indicated by

LEA guidelines, development projects, commercial publications and attainment tests) incorporate a far wider vocabulary than those of twenty years ago. This arises from a determination, on the part of those designing the scheme, to base mathematics on a broader foundation than is provided by counting and number work alone".

"There is only one way in which a teacher can find the extent of a child's number knowledge - that is by observing and questioning him, much as she hears him read. This should take place often with young children and regularly with juniors".

"The understanding of place value, with a full working appreciation of the number system and the notation we use, should be a fundamental aim with all children in the primary school".

"Mental arithmetic, especially of the kind involved in good number conversation, should not be neglected by the teacher. Questions need to be pitched at a level simple enough to ensure a very high success rate".

"The mathematician abstracts from experience, manipulates almost 'unthinkingly' and translates the symbolic result back into experience. To yield real power, he must be able to manipulate symbols easily enough to maintain his grasp of the whole process - yet this essential facility is often denigrated as 'mere' manipulation or 'mechanical' arithmetic".

"Pictorial and graphical representation is a most important aspect of mathematics. The action of recording mathematical information in pictorial form is often the first introduction to graphical work and the ability to do so is a necessary and fundamental skill well within the compass of most children".

"Mathematical Development - Primary Survey Reports 1 and 2 of the Assessment of Performance Unit".

(Published by H.M.S.O., 1980, 1981).

"Many school subjects involve mathematical skills and concepts to some degree and every part of the school curriculum depends upon and makes its own contribution to language development. In the same way scientific development is not solely the product of science teaching, nor does aesthetic awareness grow solely from art, music and drama".

"Mathematics is essentially a problem-solving activity but it is also a unique and powerful form of communication".

"The past influence of mathematics on culture and its growing current importance are reflected in the curriculum by its prominent place as a subject and by its application within nearly every other area of study in schools".

Despite the wide choice of schemes and materials which are available, there is general agreement about the broad principles on which the approach to mathematics for the 5 to 7 year-olds should be based. The work of Z P Dienes and C Gattegno, stressing the employment of concrete materials to allow children to become familiar with mathematical structures, and J Piaget's emphasis on learning with understanding and by doing, have exerted a considerable influence on British schools.

The publication, from 1964 to 1973, of Teachers' Guides by the first Nuffield Foundation Mathematics Teaching Project also encouraged the broadening of the curriculum to include practical measurement, pictorial representation, estimation, pattern, work on shape and symmetry, rather than concentration on purely arithmetical processes.

The Teachers' Guides produced by the Early Mathematical Experiences Project (1978) and the "Bronto Books" of the new Nuffield Maths 5-11 Project (1979) indicate the growing tendency to stimulate children to "talk about mathematics" so that the use of effective mathematical vocabulary is closely linked to general language development.

Flexibility in the organisation of the class permits the best use of different strategies such as teaching individuals, groups or the whole class, as and when appropriate. This flexibility takes into account the fact that children learn at different rates and so will not always reach the same stage simultaneously.

Examples of published materials

From among the materials most widely used in England and Wales this selection of four has been made to include:

- 1 example which uses expendable workbooks for pupils;
- 1 example which uses expendable worksheets;
- 1 example which uses non-expendable workcards;
- and 1 example, intended for children from 3 to 6 years old, which is based on booklets for the guidance of teachers.

There is no significance in the order in which the examples are listed; it is based on the first date of publication. A detailed list of contents for each example is contained in the Appendix of this paper. (Editor's note: Some of the materials were available for inspection during the Seminar.)

1. Mathematics for Schools (Level 1 for 5-7 year olds)

Published by Addison-Wesley

First Edition 1970, Revised Second Edition 1979

Main Authors: Arnold A Howell, Ruth Walker and Harold Fletcher

Materials: 7 separate workbooks of 60 pages each for pupils

Teachers' Resource Book (309 pages)

40 Spiritmasters providing consolidation and/or evaluation material

The aims of this series are:

to encourage children to enjoy mathematics;

to understand other aspects of mathematics as well as numeration;

to use their existing knowledge to try to solve a problem rather than to say, "We haven't done that yet";

to think logically and analytically;

to talk about and discuss the mathematics they are doing;

to become aware of the mathematics all around them.

The Teachers' Resource Book is comprehensive, giving details of the objectives, mathematical background, teaching suggestions and introductory and follow-up activities for each section contained in the pupils' workbooks. The scheme goes on to Level 11 with non-expendable texts for children from 7 to 13 years old and accompanying Teachers' Resource Books.

2. Early Mathematical Experiences (for teachers of children from 3 to 6 years old)

Published by Addison-Wesley for the Schools Council in 1978

Project Directors: Professor Geoffrey Matthews, Mrs Julia Matthews

Writers: Audrey Nicole, Jill Hancock, Margaret Adams, Yvonne Wardrop and Bob Perry

Materials: A general guide and 6 booklets for teachers covering 12 main topics (see appendix)

Videotape pack. Sequences showing children's development at six-monthly intervals, with notes

Slide pack. 30 slides with notes.

The aims of the project are:

- to identify and classify relevant experiences leading to mathematical ideas by observing young children in the classroom and relating their work to the theoretical development of early mathematical concepts;
- to produce guidance for teachers to help them to stimulate the development of mathematical concepts in young children.

To carry out the first aim it was necessary to spend some time building up a collection of experiences and conversations involving young children. E.M.E. suggests that careful thought be given to the mathematics that is inherent in any activity or material that children are offered. If a child shows a particular interest in the mathematical side or asks questions he can be guided and helped to explore the language involved. The booklets are not meant to be a complete scheme of work but are written to explain and demonstrate some of the mathematical ideas that young children need to absorb before starting mathematics in the more formal sense.

3. The Nuffield Mathematics 5-11 Project

Published by Longman Group Ltd. for the Nuffield-Chelsea Curriculum Trust

First published 1979

Project Director: Eric Albany

Authors: Wyn Moore and Monica Williams

Materials: 338 worksheets available in printed or spiritmaster form

24 Bronto Books (Authors: Pauline Burke and Wendy Dewhirst)

Teachers' Handbook 1 (5-6 year olds)

Teachers' Handbook 2 (6-7 year olds)

Videotape showing materials being used in school

The general aim of the project is to promote understanding of the concepts and proficiency in the basic skills of mathematics. For pupils at Infant level (5-7 years) the objectives are:

- to provide a wealth of activities suitable for a wide range of abilities and backgrounds;
- to link language development with the extension of mathematical vocabulary;
- to promote a gradual and confident development of mathematical understanding and skills.

The objectives of the Teachers' Handbooks are:

- to give teachers clear guidance on the content, methods and timing appropriate at each stage of the course;

- to provide practical, 'down to earth' suggestions for teaching Number, Measurement and Shape;
- to give ideas for making worksheets, wordcards, charts, models, etc., and guidance in the use of apparatus either home-made or commercially available;
- to stress the importance of language in the teaching of mathematics, for example by reference to the Project's 'Bronto' Books linking key words in mathematics with language development;
- to suggest ways of diagnosing and dealing with children's difficulties.

The scheme continues with non-expendable texts for children from 7 to 11 years old and accompanying Teachers' Handbooks.

4. Peak Mathematics

Published by Thomas Nelson in 1981

Authors: Alan Brighouse, David Gobar and Peter Patilla.

Materials: Infant Handbook 1 suggests suitable mathematical activities for 5 and 6 year olds. There are no pupil materials produced to accompany Handbook 1. Infant Handbook 2 suggests suitable mathematical activities for 6 and 7 year olds and is supported by 232 workcards on number.

Pupils' Book O, a non-expendable text concentrating on place value in the denary system.

The workcards are profusely illustrated by coloured photographs. There is a strong bias towards number work but the Handbooks do contain some suggestions for practical work in measurement and shape. The Teachers' Handbooks also contain many coloured photographs but have been criticised for not containing as much background information or as many suggestions as those of other schemes and for their heavy emphasis on skills at the expense of a deeper conceptual understanding.

The scheme continues with non-expendable texts for pupils from 7 to 12 years old and accompanying Teachers' Guides.

The examples of commercially produced materials listed above give some indication of the different formats available. Although local advisers may have personal preferences, they are prepared to offer balanced opinions on the various strengths and weaknesses of the materials. No compulsion is exercised. Headteachers are free to choose not only which materials but also how to use them.

Some infant schools devise their own basic mathematics programme and select those sections of published schemes which are supportive. Most infant schools, however, tend to base their programme on one published scheme and supplement it either with their own ideas or with parts of other published materials.

It is widely believed that the freedom of choice from a range of materials encompassing different approaches and emphases is responsible for the extremely health state of Infant schools in Britain and the enthusiastic commitment of their teachers.

APPENDIX - CONTENTS OF SELECTED SCHEMES

Mathematics for Schools (Level 1 for 5 - 7 year olds)Pupils' Workbook 1

Introduction to Sets
 Sets and Subsets
 Solid Shapes
 Plane Shapes
 Pattern
 The Idea of Matching

Pupils' Workbook 2

Cardinal Number; two, four, three,
 five and one
 Ordering the Cardinal Numbers:
 one to five
 Cardinal Number and Sequential
 Patterns; zero and six to ten.

Pupils' Workbook 3

Cardinal Number; one to ten
 Introduction to Addition
 Solid Shapes
 Addition
 Pre-measurement Relations
 heavier, lighter, larger,
 smaller, heaviest, lightest, etc.
 taller, shorter, longer, higher,
 lower, holds more/less, tallest,
 shortest, etc.

Pupils' Workbook 4

Measurement: Length
 Addition
 Comparison: "take away",
 "adding on"
 Introducing Two-digit Numbers
 Measurement: Capacity and Mass
 Counting On

Pupils' Workbook 5

Counting On and Addition
 Counting Back and "Take Away"
 Addition, Difference and "Take Away",
 Coin Recognition and Exchange
 "Sets Of"
 Enrichment Activities

Pupils' Workbook 6

Sharing
 Algebraic Relations: Open Sentences
 and Truth Sets
 Measurement: Height and Length
 Addition Using the Associative
 Property; Money
 Measurement: Time, Mass, Capacity
 Counting on Patterns
 Solids: Volume
 Enrichment Activities

Pupils' Workbook 7

Measurement: Time
 Computation
 Solid and Plane Shapes
 Algebraic Relations
 Plane Shapes
 Algebraic Relations: Inequalities
 Plane Shapes
 Algebraic Relations: Open
 Sentences and Truth Sets
 Symmetry
 Enrichment Activities

Early Mathematical Experiences ProjectTeachers' Booklets

General Guide

Space and Shape
Comparisons

Home Corner
The Family

Water
Raw Materials

Towards Number
Apparatus, Toys and Games

Outdoor Activities
The Environment

The Passage of Time
Rhymes and stories

Nuffield Maths 5-11 Project Teachers' Handbook 1 (5-6 year olds)Sets and relations (N1)

Relations (different types of correspondence)
Early sorting experiences - leading to sets
Sorting into subsets

Matching (N2)

Matching to find equivalent sets
Two non-equivalent sets
Three non-equivalent sets
Pictorial representation and semi-tallying

Counting and numerals (N3)

Counting
Matching a number symbol to a set
Introducing number words
The empty set
Conservation of number

Ordering (N4)

Ordinal numbers
Putting non-equivalent sets in order
Signs "is greater than", "is less than"
Tallying and pictorial representation

Towards addition (N5)

Composition of small numbers
Addition of two disjoint sets
Recording addition by mapping

Addition to 10 (N6)

Number bonds up to 10
Counting on
Patterns in simple addition
Picture problems - additions
Introduction of addition sign and vertical addition
The addition square

Length (L1)

Descriptive language
Comparing two unequal lengths
Matching lengths: matching two objects of about the same length
Matching lengths: using several objects to 'make up' a length
Ordering
Measuring with repeated units
Using limb measures

Shape and space (S1)

Awareness of shape and space, extension of vocabulary
Early sorting of 3-D shapes
Early sorting of 2-D shapes

Weighing (W1)

Descriptive language - heavy and light
Comparing - heavier than and lighter than
Balancing
Ordering

Time (T1)

Matching events to daytime or night-time
Putting time sequence in order
Comparisons: fast and slow
Comparisons: timing
Graphs and charts

Money (M1)

Recognition of 1p, 2p, 5p, 10p coins
Comparison of amounts of money
(a) by matching (b) by totals
Using coins to make amounts up to 10p
Early stages of shopping
Suggestions for shops in schools

Capacity (C1)

Full, empty and half empty
Which holds more? (visual judgement)
Finding the capacity by counting
Sorting containers
Comparing by emptying
Comparing by filling

Nuffield Maths 5-11 Project Teachers' Handbook 2 (6-7 year olds)

Early stages of subtraction (N7)

Finding the difference
Counting back
Taking away
Recording subtraction
Practice sheets

A first look at place value (N8)

Early group activities and games
Grouping using cubes, etc.
Grouping in tens

Addition to 20 (N9)

Number bonds up to 20
Counting on
Ways of recording

Subtraction involving numbers up to 20 (N10)

Difference by matching and counting
Subtraction by counting back
Taking away
Ways of recording

Introducing multiplication (N11)

Recognising and counting equivalent sets
Multiplication as repeated addition
Arrays and the commutative law
Activities and games for 'table facts' up to 30

Introducing division (N12)

The sharing aspect of division
The repeated subtraction aspect of division
Division as the inverse of multiplication
Remainders

Length (L2)

Appreciating the need for a standard measure
Introduction of the metre
Comparison with a 10 cm rod (decimetre)
Measuring in centimetres - straight and curved lines
Personal measurements in m and cm

Shape and Space (S2)

Sorting for shape and size
Fitting shapes together
Surfaces and faces
Covering surfaces - leading to area
First ideas of symmetry

Weighing (W2)

Introduction of kilogram and $\frac{1}{2}$ kilogram
Using the kilogram and $\frac{1}{2}$ kilogram
Introduction of the 100 gram weight

Time (T2)

Ways of measuring time
Reading a dial
Telling the time (hours, halves, quarters)
Telling the time (five-minute intervals)
Simple calculations involving time
Other units of time

Money (M2)

Reinforcement of coins up to 10p and introduction of 50p
Breakdown of coins - equivalent value
Making amounts up to 20p
Addition - simple shopping bills
Giving change and finding difference by counting on
Subtraction by taking away

Capacity (C2)

Introduction of the litre
Comparing a litre with non-standard measure
Introduction of $\frac{1}{2}$ litre and $\frac{1}{4}$ litre
Cubes, boxes and walls

Peak Mathematics Infants Handbooks 1 and 2 (5-7 year olds)

Section 1

Early Mathematical Activities Sorting activities, sequencing activities, comparison activities, ordering activities, one-to-one correspondence, number line activities.

Section 2	Section 3	Section 4
<u>Numbers up to 10</u> Oral counting Number symbols Sequencing Numbers as words Writing of number symbols Number groups	<u>Numbers up to 10</u> Recognition of number groups One-to-one correspondence Seriation Combining groups (addition) Partition groups Subtraction Complementary addition Addition - various formats Subtraction - various formats	<u>Numbers up to 20</u> Addition Subtraction Difference Addition and subtraction in various formats Early ideas on multiplication Early ideas on division
Number line activities up to 20	Number line activities up to 50	Number line activities up to and over 100
<u>Shape</u> Observation and discussion of natural and geometric shapes	Recognition of similar plane shapes and solid shapes	Revision of plane and solid shapes
<u>Measurement</u> Length - ordering activities Weight - comparison balancing activities arbitrary units	Length Activities with Weight directed use of Capacity arbitrary units	Length Activities Weight involving a Capacity choice of arbitrary units
<u>Money</u> Recognition and naming of coins	Simple addition of coins to 10p Equivalence of coins to 10p	Equivalence of coins to 50p Addition and subtraction of coins to 50p
<u>Time</u> Functional times within the day Days of the week Months of the year	General calendar work, including the seasons Weather chart	Telling the time
<u>Graphs</u> Arrow graphs	Picture graphs	Picture graphs

At the Oxford Seminar and in preparation for this second Seminar, the Soviet participants had pointed out that the school starting age in the Soviet Union was being lowered from 7 to 6 years. They were therefore particularly interested in the British experience of

teaching mathematics in schools at the infant level. Accordingly, the considerable detail of four major schemes given in Mr Albany's paper was appreciated, and great interest was generated by the examples of materials that were circulated and were on display.

In his presentation, Mr Albany underlined the great diversity of materials available to an English school at the Infant level. Two major aspects of teaching mathematics to these young children are an emphasis on activity and an emphasis on the development of language in close conjunction with mathematical activity. Children were encouraged to talk about what they were doing, and teachers did not push them into written recording too soon. Indeed, it is impossible to give a universal answer to the question when the transition is made to written recording, beyond saying that it is when individual children are ready.

A wide variety of other material, both manufactured and from the environment, are used in early mathematics teaching in English schools; for example, much can be done with a collection of leaves, for example sorting by various criteria, symmetry and early ideas of similarity and area.

THE CONTENT AND METHODS OF MATHEMATICS TEACHING IN PRIMARY SCHOOL

Professor A M Pyshkalo, Head of the Primary Teaching Laboratory, Institute for Scientific Research into Content and Methods of Instruction

The first three years of learning (classes I-III) constitute the primary stage of the Soviet general school. Class I accepts children who by the first of September have reached the age of 7. In each of classes I-III there is one mathematics lesson per day. Thus considerable attention is given to the study of this subject: 25% of total teaching time.

Over the last 15 years the mathematics course for classes I-III has undergone considerable changes, the essence of which was made clear to participants in the 1981 England-USSR symposium (see paper by A M Pyshkalo and V V Firsov).

An undoubted achievement in the development of the content of mathematics teaching for younger pupils is that its traditional isolation from the content of mathematics teaching for class IV and above has been overcome. It may be said that the primary mathematics course is now being formed into an organic part of a single secondary (classes I-X) school mathematics course, since it carries on a significant share of the work on arithmetical operations on natural numbers and their applications to simple magnitudes, in the visual introduction of individual geometric figures and their properties. The central (basic) concepts of this course are still the concepts of number and magnitude.

In the process of forming children's ideas (pre-concepts) of the whole, non-negative number, an important role is played by the study of the four arithmetic operations - addition and subtraction (beginning in class I), multiplication and division (classes II and III), together with their most important properties. The study of these properties, as an analysis of widespread educational practice shows, leads to a more conscious mastery of the methods of calculation and somewhat lessens the amount of time which must be spent on teaching calculation.

The teaching of the properties of the arithmetical operations to younger children and the use of these properties in the process of learning the methods and skills of calculation is undoubtedly a progressive step which makes it possible not only to enrich the mathematical content of the primary course and to ensure greater continuity with the subsequent course content but also to increase somewhat the role of theoretical generalisations (theoretical knowledge) in mathematics teaching in the first stage of schooling. The latter circumstance is also very important when one comes to evaluate changing teaching methods.

We have no doubt that both the study of methods of calculation and the use of the most important properties of the arithmetical operations in this process are important not only for teaching pupils particular algorithms of calculation but also for children's general development.

However, the content of and methods for this work are still beset with problems which have been insufficiently studied. The core of these problems, in our view, is the search for an optimal relationship between, on the one hand, the practical and the visual, and on the other the abstract and the formal in the teaching and application of the major properties of the arithmetical operations and, more importantly, of their consequences.

There are two separate approaches to the solution of this problem. The first is the "direct" "frontal" approach to the study and use of these properties, beginning with the first class, at the same time as the methods and algorithms for carrying out the operations are learned (M A Bantova, M M Moro, A S Pchelko and others). In this case, inevitably, the pupils must learn by heart certain rules and writing conventions. The realisation of this method in practice has shown that although such a "direct" application of theoretical knowledge is possible, it is associated with a number of didactic difficulties in the case of many children, and as yet no effective or correct means have been found for overcoming these difficulties. It is not simply that in this case learning is associated with a heavy overloading of the memory, that given this approach children must assimilate a large number of difficult and abstract "rules" for adding a number to a sum, a sum to a sum, subtracting a sum from a number etc, that in addition it is necessary to learn to carry out a corresponding set of equally cumbersome pieces of writing; there is also the point that it becomes necessary to introduce a number of "restrictions" to the use of the "rules", as a result of which the rules lose their unity, their essence - and this is something which many pupils discover quite quickly (especially those who can already compute well), for these restrictions contradict their experience.

Thus, for example, when two two-digit numbers are added together the rule for the addition of a sum to a sum is used, but only in the case where the first item of the second sum is added to the first item of the first sum and the second item of the second sum is added to the second item of the first sum (none of the other possible cases is considered). Thus this "rule" is used only as follows:

$$21 + 34 = (20 + 1) + (30 + 4) = (20 + 30) + (1 + 4) = 50 + 5 = 55$$

It is not difficult to imagine the difficulties to which such a treatment of the rule in question leads in subsequent teaching.

The second approach (K I Neshkov, V N Rudnitskaya and others), takes into account the difficulties and disadvantages of the first approach and makes its starting point the mastery of the basic properties of, say, addition, but with the necessary condition that pupils should use the concrete operations in conjunction with real objects (coloured sticks, counters, the scale of a ruler etc.) This approach, which emerged alongside the development of a school mathematics course commencing from the age of 6, envisages the earlier introduction of written calculations on the basis of a sound grasp of addition tables (class I) and multiplication tables (class II) of single digit numbers. So-called "number structure" is here studied on the basis of already well-known addition.

The second approach envisages closer interconnections in the study of magnitude and number. Children learn to measure magnitude - length, mass, time, area. Here the greatest attention is devoted to the development of measuring skills, the study of the scales of corresponding instruments of measurement, the use of these scales in the solution of practical problems, mastery of the most common units of magnitude. Great attention is given to the study of dependence between magnitudes, a knowledge of which is used by pupils in the solution of elementary problems.

The second approach also differs somewhat from the first as regards the inclusion of algebraic material in the course for younger pupils, in the direction that it follows and the ends to which the study of it is put. Although the first approach does include moderate amounts of algebraic material, it is used primarily to "fix" those generalisations which emerged from the study of the properties of the operations (for example $a + b = b + a$; $ab = ba$ etc), and in the solving of very simple equations of the type $x + 2 = 5$ (class I), $2x + 1 = 7$ (class III) on the basis of knowledge of the dependences between components of the operations and their result. We note that even in this case there is a heavy overloading of the memory: it is necessary to know six rules for finding the unknown component. Here we see an unwarranted "foray" into the study of algebraic material, which is connected with the forthcoming appearance of the equation as a means of solving problems; this is particularly noticeable when simple problems, the solution of which is perfectly straightforward in the normal way, are solved by forming equations, causing difficulties for many children.

The second approach attempts to avoid the above-mentioned disadvantages. Here the algebraic material is introduced more gradually and in a more limited way into the primary mathematics course. This is done on the basis of the generalisation of arithmetical knowledge, which all pupils have already properly mastered, and a decrease in the number of potential didactic pitfalls in the introduction of essential algebraic preliminaries. This is immediately obvious from the more measured, cautious and pedagogically well-founded manner in which the mathematical symbolization used in the teaching of children is extended.

In general one of the most important criteria governing the inclusion of this or that mathematical fact in the syllabus is not merely the criterion of accessibility, but a clear understanding of how that fact (piece of knowledge) will "work" later on in the course, how it will serve to bring about mastery of the system of mathematical knowledge covered by the whole (or the greater part) of the mathematics course. In other words, if this or that fact is not used or cannot be used for any length of time in the course of study, then in spite of its accessibility it is not included in the course content. The consistent application of these and other criteria has enabled our methodologists to avoid some well-known enthusiasms which have led to some well-known mistakes in the compilation of "modernist syllabuses".

Such is the essence of those tendencies which are determining the changes in the content and methods of teaching in primary classes. At present, on the basis of a study of the working experiences of many teachers, they are incorporated in "improved syllabuses" within the system of the knowledge, abilities and skills which all pupils must have acquired by the time they reach the end of each year's work.

Pupils of class I must:

know the table of addition of one-digit numbers and corresponding cases of subtraction;

- be able to
- read and write numbers from 10 to 100;
 - find (by mental arithmetic) the sum and difference of numbers under 100, in cases of the type $34 + 6$; $98 - 5$; $30 + 4$; $50 + 20$; $70 - 4$; $80 - 36$; $24 + 30$; $78 - 20$; $35 + 14$; $86 - 23$;
 - measure the length of a line in centimetres and decimetres and draw (with the help of a ruler) lines of given length;
 - identify and name a line-segment, polygons;
 - solve problems involving addition and subtraction in one operation.

Pupils of class II must:

know multiplication tables for one digit numbers and corresponding cases of division;

- the order of carrying out operations in expressions containing 2 or 3 operations;

- be able to
- read and write numbers up to 1000; carry out with precision addition, subtraction, multiplication and division within the limit of 100;

- solve very simple equations;
- solve problems involving 1 or 2 operations (by making up expressions or operation by operation);
- find the value of simple expressions for given values of the letters contained in them;
- identify, reproduce and designate with letters a point, a line, a segment of a line, a polygon;
- find the perimeter of a polygon, including a rectangle (square).

Pupils of class III must:

know by heart the table of addition for one digit numbers and corresponding cases of subtraction, and the table of multiplication of one digit numbers and corresponding cases of division;

- the order for carrying out operations in expressions containing 3 or 4 operations;
- the units of length, mass, area and time and the relations between them;

be able to read and write multi-digit numbers under a million;

- carry out operations within the limit of million, and to do so mentally within the limit of 100;
- check to correctness of calculations;
- read and write simple expressions and to find their numerical value;
- draw a rectangle (square) on graph paper, with given length of sides;
- find the area of a rectangle (square);
- solve simple problems involving 1, 2 or 3 operations.

A more accurate assessment of the cognitive capabilities of younger pupils and the indicated changes in the course content have had a certain influence on teaching method and in particular on the structure of the mathematics lesson. Whereas previously in primary teaching there were lessons which were devoted entirely to the communication of new material (or entirely to consolidation, or the revision of what had been learned), now the basic type of lesson is multi-purpose in that it includes both the communication of new material and prior consolidation and revision.

Whereas previously the basic method of communicating new knowledge was exposition (telling) on the part of the teacher, while the children memorised and then on the teacher's request reproduced that which the teacher had said, now discussion is being used more and more for this purpose - discussion in which pupils play an active part. Also, discussion is linked with independent work, for which work-cards and work-books have been produced. These methods are aimed at organising productive and creative work which is within the capabilities of the pupils. The carrying out of various types of practical work using real objects is occupying an increasingly important position at all stages of the acquisition of mathematical knowledge, ability and skills. A larger role is being played in the study of mathematics by games and play-techniques, which provide pupils with tremendous opportunities for the conscious and automatic use of the knowledge they have acquired and for the acquisition of new facts.

Unfortunately, Professor Pyshkalo was unable to attend the Seminar, and the paper was not formally presented. It is included in the report, however, as a clear description of the Soviet primary mathematics curriculum, complementing the paper by Mr Albany.

CONTENTS AND METHODS IN THE TEACHING OF ALGEBRA AND BASIC CALCULUS

J N Makarychev, Institute for Research into the Content and Methods of Instruction

1. In the existing mathematics course pupils begin to become acquainted with elements of algebra in classes 4-5. In these classes letter symbols are introduced, the concept of an expression with variables is covered, and simple transformations of expressions are examined (reduction of similar members, expanding brackets and factorising, taking the common factor outside brackets). Here the emphasis is on linear expressions. Pupils become acquainted with the concept of the equation and solve equations of the first degree with one variable and use equations to solve simple problems. Also in these classes the concept of a set is introduced, along with that of an element of a set and the operations of union and intersection of sets, and the simplest logical concepts are brought in: statement, variable, proposition with variables. In this way in classes 4-5 the essential foundations are laid for the study of the systematic algebra course which begins in class 6.

2. Before the reforms of the 70's the study of algebra began in the second half of the 6th year. At that stage the training that pupils were receiving was of an exclusively arithmetical nature, limited moreover to a study of the arithmetic operations on positive rational numbers alone (negative numbers were introduced into the algebra course in class 6). No algebraic symbols were used, problems were solved by arithmetic means and the method using equations was absent. This seriously delayed pupils' development and created problems for the early lessons in algebra, where the transition from arithmetic to algebra was too sudden. This meant that a considerable amount of time was required for the introduction and study of initial algebraic concepts. As a result the study of algebra in 8-year school was limited to transformations of rational algebraic expressions and first degree roots, and the solution of equations and simultaneous equations with two unknowns of the first and second degrees. The concept of function was not introduced until class 8 and was not sufficiently developed. In classes 9-10 inequalities, powers with fractional indices, arithmetic and geometric progressions, exponential and logarithmic functions, trigonometric functions and their corresponding formulae were studied. Therefore pupils in secondary school never became acquainted with topics of great general-educational importance such as the derivative and the integral.

3. The reform of the 70's was characterised as regards the study of arithmetic-algebraic material by the fact that in the syllabus a stable core of the course was preserved. At the same time the syllabus was based on the successive development of

mathematical knowledge from classes 4-10 inclusive. The mathematics courses for years 4-5, the algebra course for 6-8 and the algebra and basic calculus course for 9-10 were linked by their ideas. Topics such as inequalities, arithmetic and geometric progressions, the exponential function and decimal logarithms were transferred to the algebra course from the upper classes. The function line of the course was considerably strengthened. The general concept of the function and functional symbolisation began to be introduced in the algebra course for the 6th class. This made it possible for ideas about the graphs of functions to become well established in 8-year school, which in turn made possible the introduction of elements of mathematical analysis in the senior classes.

4. In school algebra there are four separate "threads" to the content; rational and irrational numbers, the identical transformation of expressions, equations and inequalities, and functions and their graphs. Each thread is developed in sequence throughout the course, covering the entire period during which mathematics and basic calculus is taught, and at the same time the development of each of the threads is closely linked to that of the others. This is possible because of the fact that school algebra rests on a comparatively small number of fairly common concepts. Their systematic use throughout the course makes it possible to unify the language used and thereby to facilitate the establishment of links within the subject. This is further brought about by the use of various exercises, the carrying out of which ensures the use of knowledge from various sections of the course.

5. The systematic study of rational numbers begins in class 4. Subsequently this knowledge is systematised and generalised. In the 7th class pupils acquire their first ideas of irrational number and the concept of the real number is introduced. The study of real numbers is closely tied to their representation on a co-ordinate straight line. Pupils are required to understand that between the set of real numbers and the set of points in a co-ordinate straight line there exists a simple correspondence, and using the example of square roots that it is possible to find any approximations of irrational numbers in the form of finite decimal fractions. The pupils' understanding of irrational number is made more precise subsequently, in class 9.

In the process of studying algebra and basic calculus, skill in the use of the operations on numbers is developed. This is done by means of various exercises which are systematically included in various sections.

In accordance with the amendments made to the mathematics syllabus in 1981, it is intended to acquaint pupils in the senior classes with complex numbers.

6. The transposition of expressions is studied throughout the algebra course for classes 6-8. In class 6 transformations of algebraic expressions are examined; this is extended in class 7 to include transformations of algebraic fractions and square roots, and in class 8 to include powers with rational indices and first degree roots. Special attention is devoted to the identical transformation of rational expressions, which constitute the basis of the study of series of transformations and their applications to the solution of equations and inequalities. Therefore it is more important to develop skills in the identical transformation of rational expressions than in transformations of other expressions. At the same time the syllabus allows for unwillingness to practise cumbersome and labour-consuming exercises of the sort that existed previously (before the reforms of the 70's).

Identical transformations of expressions are used in the study of various topics in the algebra and basic calculus course. In particular, in the senior classes, they are used in the study of trigonometric functions and the corresponding trigonometric formulae.

7. In the study of equations and inequalities particular attention is devoted to equations and inequalities of the first and second degree with one variable and to systems of linear equations with two variables. In the solution of simultaneous equations of second degree the systems are usually such that one of the equations is first degree. In 8-year school there is also a study of equations which contain a variable in the denominator of a fraction and which reduce to linear or quadratic equations. Also, the simplest exponential and logarithmic equations are studied. In the senior classes simple trigonometric equations and equalities are studied. Here the greatest attention is devoted to equations of the type $\sin x = a$, $\cos x = a$, $\tan x = a$. Equations of the first and second degree with one variable and simultaneous equations with two variables are used widely in the solution of problems. Theorems about the equivalence of equations and inequalities are not studied in 8-year school. This material is taught in the senior classes.

In the study of equations, inequalities and simultaneous equations, widespread use is made of the Cartesian form of the equation of a line, the co-ordinate plane and corresponding graphs.

8. The concept of the function is introduced for the first time in class 6. The concepts of the range of definition and the range of values of a function are taught at the same time, and functional symbolisation is introduced. In the algebra course for classes 7-8 a study is made of the increment and 'decrement' of functions and the concept of the inverse of a given function is taught. The introduction of these concepts is linked with the study of a number of actual functions: $y = kx + b$, $y = \frac{k}{x}$, $y = ax^2 + bx + c$, $y = x^n$, $y = \sqrt[n]{x}$, $y = a^x$, $y = \log x$. In the algebra for classes 6-8 emphasis is placed on pupils' graphical

presentations. The visual quality of the graph makes it possible to examine on an accessible level the various properties of functions, and to make the transition from geometric conceptions to the corresponding analytic definitions. The graph of a function represents a key image with the help of which the essential formalisation of the material being studied is achieved. The systematic use of functional apparatus in the study of other algebraic topics leads to a better mastery of the concept of the function and other course topics.

The work on functions which takes place in 8-year school facilitates the successful study of elements of differential and integral calculus in classes 9-10. Here the emphasis is on the concepts of the derivative and the integral. Topics connected with the concepts of limit and continuous functions are used for back-up purposes. The concept of the derivative is used in the construction of graphs of functions and the solution of various maximum and minimum problems. The definite integral is used in the solution of simple geometric and physical problems.

In the algebra and basic calculus course the types of functions and their properties which were studied previously are now treated more fully. Functions with powers, logarithmic, trigonometric and inverse trigonometric functions are covered. The concept of the periodicity of functions is taught.

In the algebra and basic calculus course the knowledge acquired by pupils through the study of algebra in 8-year school is brought to its final level of systematisation and generalisation.

9. The algebra course for classes 6-8 is primarily inductive in structure. Unlike geometry, no system of axioms is taught, and some concepts are introduced without formal definitions. However the proofs of individual facts are covered by the course. This applies to the deduction of certain algebraic identities, to the finding of formulae for solving equations, to the study of functions by elementary means and to the methods of solving various problems. As the pupils progress from class to class the role of the deductive method of reasoning increased. Thus, although in the study of functions the greatest attention is devoted to mastery of the properties of functions on the basis of graphic representation, and the ability to read the properties of the functions from the graph, at the same time deductive reasoning and the corresponding analytical computations begin to be included, either in the process of constructing the graph or previous to its construction. In classes 9-10 the role of deduction increases.

Throughout the teaching of algebra, great importance is attached to preparation for the formalisation of the material being studied. This is evident from the use of visual images (graphs), the examination of numbers of similar examples with a view to making generalisations, and the concretisation of methods of proof.

10. In the study of algebra great importance is attached to the disclosure of the content of the concepts being studied and to acquainting pupils with the most important ways and methods of using the concept in the course. As a rule the formation of concepts is achieved on the basis of pupils' concrete experience and is accompanied by an examination of visual examples which help to clarify the concept and by the adduction of counter examples.

Showing pupils the applications of a new concept directly in the course itself is an important psychological factor which increases pupils' interest in what they are studying and convinces them of the usefulness of the new concept; in particular, of the fact that with its help various types of problems can be solved. This trend in the teaching of algebra facilitates the intensification of the applied tendency in the teaching process.

11. One of the problems which must be solved in the teaching of algebra and basic calculus is that of developing sound skills in the transposition of expressions, in the solving of equations and inequalities, and in the construction and reading of graphs and functions. This problem, of how to develop sound skills, is solved by means of the use of sufficiently simple exercises.

12. The aim of mathematics education and of the study of algebra in particular consists not only in the mastery of particular knowledge and skills but also in the formation of certain qualities which characterise a pupil's mathematical development. Teaching which encourages mathematical development results from the use of exercises with varied content, which permit various methods of solution including non-standard solutions, and from acquainting pupils with supplementary knowledge of methods of solving problems and proving theorems.

This paper, dealing with two of the more abstract themes of the school mathematics curriculum, led to a discussion of a number of fundamental issues on the construction of that curriculum.

The English participants asked for clarification of how teaching groups are arranged in schools. Is there ever any setting for mathematics? No. However, some optional extra work, in the form of harder examples, can be made available in Classes 4-6. There are

also mathematical clubs in schools, where non-standard work can be explored, and it is sometimes found that pupils who are slower in mathematics in class react favourably to these non-standard tasks. Within the formal curriculum, however, all pupils do the same, including those in special schools for which general educational achievements are a minimum requirement. This policy follows directly from the basic principles of the Soviet school system.

Liaison between the algebra developed in the mathematics course and the needs of other subjects is achieved by the fact that curricula are determined by two commissions, one of which is subject-specific, the other an overall Curriculum Committee. Changes must be approved by both, and the latter Commission's function is to ensure such necessary coordination. This is easier to achieve in a centralised system.

In England there is a problem at the beginning of algebra of assigning meaning to symbols. Pupils need to get used to using symbolism in simple situations before getting involved in algebraic algorithms. In the Soviet Union there is preparatory work, including the introduction of letters for numbers, in Classes 4 and 5, whereas in the earlier curriculum algebra started from scratch in the middle of Class 6. This propaedeutic work is proving successful, and should be seen as an integral part of mathematics, rather than just as pre-mathematics. Similarly, the concept of 'variable', which the CSMS research in Britain has shown to be difficult for children to learn, is introduced in the Soviet Union in Class 4, and this technique works very well.

The British team said that our experimentation with the use of calculators is showing new ways of developing the different concepts of variable and of operation. This suggestion did not meet with approval from the Soviet side, who doubted whether the calculator could contribute to the teaching of algebra.

Session 9. Methods of Assessment in Mathematics Education

EXAMINATIONS AND ASSESSMENT

Dr Alan Bell, Shell Centre for Mathematical Education, University of Nottingham

Introduction

This topic covers the evaluative procedures used in the public examinations taken by school pupils (the GCE and CSE exams), the periodic testing of pupils' progress within the school, and also researches and surveys which seek to evaluate the success of curricula or of schools; these may be general national surveys, such as that of the APU, or smaller research projects evaluating special curricula.

We shall need to consider the purposes of these evaluations, and also the type of attainment being assessed; for example, there has been increasing differentiation between the assessment of skills, conceptual understanding and general strategies. This paper will focus particularly on this latter aspect, but we shall first review the different purposes for which evaluations are made.

Purposes

Evaluations are conducted for three main purposes: (a) Persuasion, (b) Prediction, (c) Diagnosis.

(a) Persuasion may be seen operating, at the individual level, when a teacher marks pupils' homework in order to encourage them to do it. On the public scale it occurs when material is included in examinations in order to encourage schools and teachers to devote attention to it. Even when examination syllabuses are determined by committees of teachers, they exert a strong influence towards securing the conformity of all teachers to the curriculum choices of the committee.

Conversely, what is not examined is unlikely to be learnt or taught; this implies a need for maintaining breadth in the syllabus in order not to discourage work towards desirable educational outcomes which are not easily assessed.

(b) Prediction Evaluation is used both publicly and within schools to select individuals for different courses or careers. For this purpose predictive validity is the criterion for choice of material; breadth is not important in itself, but there may still be a clash

between easily assessable material and that which effectively predicts future performance.

(c) Diagnosis Examples of assessment activities in which this is a major purpose include the British national mathematics surveys by the Assessment of Performance Unit (APU), some in-school tests, and much recently developed research material. These aim to identify particular aspects of strength or weakness in the performance of individuals and of the national educational system as a whole. Here the primary requirement is the depth of probing. As well as the facts of strength or weakness, the underlying reasons or tendencies need to be identified, if diagnosis is to lead to effective action.

The Nature of What is Being Assessed

Many of the most interesting developments in assessment procedures in recent years have involved broadening the range of attributes being sampled. In particular, distinctions between (1) facts and skills, (2) conceptual understanding, (3) general problem-solving strategies and (4) attitudes and appreciation have become recognised as important. These developments have taken somewhat different forms in the public examination system, in national surveys, in tests produced for school use, and in research. In public examinations the trend has been mainly towards separating out the testing of direct recall of facts and skills in very short items, from longer items aimed at evaluating conceptual understanding, as shown by the ability to select and use relevant knowledge and skill in appropriate contexts. These longer items also present some degree of 'strategic' or problem solving demand. However, a second trend, not on a large scale but perhaps significant, has been towards incorporating an assessment of pupils' abilities to carry through much larger scale tasks. These have consisted of a single-investigation paper, and of the development of schemes for assessing pieces of extended work some during the course.

Developments in the APU national survey have included (a) the attempt to diagnose more deeply the conceptual understanding of pupils, by using clusters of closely related items, and by interview testing; and (b) the inclusion of material testing strategies of problem solving, problem formulating, generalisation and reasoning.

Some material for use in schools by the Inner London Education Authority (ILEA) has been based on short teacher-interviews to probe particular concepts.

In the sphere of research some written test items to evaluate the mathematical strategies of generalisation, representation and proof have been developed by our Nottingham research group.

Some examples of each of these developments will now be given.

Short and Long Items in Public Examinations

An example of the trend in public examinations towards separate papers (or sections of papers) containing shorter and longer items is provided by the SMP O level papers for June 1980 (these papers were available at the Seminar). Here paper 1, section A consists of items which require essentially one-word or one-number answers, or else the choice of one from four offered answers. These are tests of specific concepts or skills, in the form in which they have been learnt; there is no demand for transfer to a fresh situation. Thus candidates are asked to perform calculations, to solve simple equations, to simplify an algebraic formula, to find the value of an angle in a right angled triangle with given integer sides, to state the probability of two dice showing the same number. This paper contains 22 questions and represents about half of the marks available.

Section B of the same paper contains five questions, each on a single situation but containing questions on several different aspects of it. Each part consists of a straightforward application of a learnt technique; the main differences from the section A questions are (a) that the candidate has to operate on elements which he has generated himself in previous parts of the question; and (b) that internal consistency can provide a check of correctness. Thus the candidate is placed in a more realistic problem solving situation. For example, in question B1, a matrix transformation has to be applied to a figure which has been plotted, and the square of the matrix has to be calculated and then interpreted as a transformation; here the interpretation may be made both from recognising the form of the matrix and from the geometry.

Thus, in section A one is testing what skills and knowledge are available on immediate recall; but in section B one can evaluate the wider range of skills which can be brought into play in a more facilitating situation in which some familiarity with the context is developed during the time spent on the problem. It is also possible in these more complex situations to test mathematical reasoning required to relate different aspects of the question. Thus, of the three types of functioning mentioned above, facts and skills are tested predominantly in section A, while section B requires some degree of conceptual transfer, and of reasoning.

Evaluating Extended Investigations

Some teachers feel that the domination of classroom work by examination-type exercises hinders the participation by pupils in mathematical activities on a scale commensurate

with their intellectual powers and their creative abilities. These exercises, they feel, compare unfavourably with the essay and story-writing and the appreciation of literature which form part of education in the humanities.

A development in the public examination field which goes very much farther in testing candidates' ability to carry through extended investigation of a situation is provided by the papers for the Mathematics Applicable project (now, unfortunately, no longer being offered). This examination, for candidates studying mathematics beyond O level, in paper I tests standard techniques, but in more realistic contexts than is usual, and in paper II offers a single situation, and leads the candidate step by step through an extensive investigation. A higher level of problem solving demand is achieved by providing hints which may be exposed (by tearing off a covering strip of paper) if needed; a penalty mark is incurred. (Sample papers from this project were available at the Seminar).

Being led through an extended investigation in the examination room falls some way short of 'normal' mathematical problem-investigation. A more common approach to evaluating this kind of activity has been to include an assessment of larger-scale work, done during the school course, into the examination procedures. This has been done only on a small scale in mathematics examinations; it is less uncommon in other subjects. It generally involves the adoption of a scheme for assessment under a number of headings. One in use as part of an O level examination uses the following headings:

1. Personal contribution and use of sources.
2. The write-up as a communication.
3. Significance of the problem, quality of problem-formation, or of design of the experiment, including extensions or generalisations in the course of the work.
4. Execution of the task as formulated.
5. Evaluation and interpretation of results; awareness of their significance.

Under each heading criteria are given for the award of different scores. For example, under the heading 'Execution of the task as formulated: A) Mathematical level and quality', the highest of the four grades is for "uses highly suitable concepts and methods...; develops concepts and methods in the course of the work". A lower grade is "feasible but not strictly appropriate methods".

Here mention should also be made of an experiment conducted by the Scottish Certificate of Education Examination Board (1975) which investigated different ways of marking project work in sixth form physics. Previously low correlations between the assessment of the project work and a written paper led the Board to consider a variety of marking

schemes and moderation procedures for the project work, involving three categories of marker: the class teacher, one of the Board's regular moderators (normally from a university), and a neighbouring teacher who was himself submitting candidates. It was found that the most successful scheme was one in which the class and neighbouring teachers were required jointly to produce an assessment, coupled with overall scrutiny by an external examiner.

Diagnosis Using Item Clusters

The national surveys of the Assessment of Performance Unit have developed several new features in their examining. Again they have in general tested (a) skills involving little or no transfer, (b) concepts and applications requiring near transfer, (c) problem solving strategies, and (d) attitudes, in distinct ways.

For purely evaluative purposes, such as determining trends in mathematical performance over time, the breadth of coverage of the curriculum would be the most important criterion. But the diagnostic value of these tests has been much increased by the inclusion in each survey of a number of clusters of closely related items, so that by comparing facility levels it is possible to judge what features of the items cause difficulty. Some such comparisons were illustrated in my paper for the 1981 seminar (see report on the first Seminar p 57). A study of several items from a cluster on ratio and proportion, used in the 1979 survey of 15 year olds (available at the Seminar), shows how the difficulty level is most strongly affected by the complexity of the ratio involved, whether a simple unitary one (4:1 or 5:1), the ratio of small integers, such as 5:2, or involving fractions ($5/6:3/4$, $1\frac{1}{2}:5\frac{1}{2}$). Two questions of each of these three types had facilities as follows: (77%, 76%; 53%, 56% and 48%, 41%; 45%, 37%). Contextual differences appeared to have little effect, though all were chosen to be relatively familiar. The implication is that when teaching proportion one cannot assume that a single general method will be applied with equal success - or even applied in the same way - to problems containing ratios of different numerical complexity.

'Practical' Interview Testing

The diagnostic value as well as the evaluative validity of the APU survey was also increased by the inclusion of 'practical' interview tests. Items from these were quoted in my 1981 paper. The value of these tests is supported by an interesting feasibility study carried by Sumner (1975), who compared children's performances on tasks that were presented in a practical and in a written mode. For the practical items children were

interviewed individually, whereas the written items were presented in the form of a class test. The written items tended to be more reliable (in the sense that children's responses were more consistent across items) and Sumner argues that the correlation of children's performances were too low to regard the two modes as equivalent. Sumner also compared the effect of presenting a written test with the same test administered orally. As can be seen from the table below, the oral version was substantially easier for 11 year olds with below-average reading ability, although the alternative forms of administration made little difference for the better readers.

Score groups of 11+ pupils above/below the median for Reading Comprehension (NFER Reading Test DE)

		RC score < 50%	RC score > 50%
Oral	Number of children	30	30
	Mean score	19.6	25.9
Written	Number of children	44	44
	Mean score	14.6	24.8
		Significance of difference Not significant between mean scores at 1% level	

Attitude Testing

Testing of attitudes by the APU has included (1) a traditional Likert scale of 34 statements concerning the Utility, Difficulty and Enjoyment of mathematics as a whole; (2) a rating given by the pupils of the usefulness and difficulty of each of a list of 17 paired topics and activities from the mathematics curriculum, and (3) a set of 15 problems chosen from the written tests, to be first solved and then rated for difficulty and usefulness. (Some examples from these tests were shown at the Seminar).

Interview Tests for School Use

Diagnostic tests intended for school use have been developed by the ILEA ('Checkpoints'), the CSMS project (Hart, 1981), and also by my research group at Nottingham. These last have been designed following diagnostic interviews which exposed a number of deep-lying

misconceptions, and they form part of a package with teaching materials designed to remedy the misconceptions. The marking scheme is designed so that the teacher can fairly easily identify the misconceptions commonly held in his class, and can select from the teaching material those activities and exercises which are most appropriate. (Samples were available at the Seminar).

The ILEA Checkpoints form a comprehensive scheme for diagnosis, remedial help and the recording of progress in the primary school. (A page from the record sheet and specimen card were available at the Seminar). The essence of this system is the observation by the teacher of the method being used by the pupil for the problem. But note that 'background information' is also provided to ensure that the teacher understands fully the purpose and design of the probe.

Written Tests of General Mathematical Strategies

Some attempts have also been made to assess strategies directly. Items aimed at testing abilities to generalise, to explain or prove, and to use symbolic or diagrammatic representations have been developed by one of our research projects at Nottingham. (One of the tests was available at the Seminar). Some items will be discussed here - the item called 'Stamps' is as follows:

"Suppose you have a lot of stamps of value 5p and 7p but no others. You can make up various amounts of postage from these. For example, you can make up 17p as $7p + 5p + 5p$.

Jane says 'You can't make up an amount of 13p'.

Explain fully why Jane is right."

It aims to test an aspect of explanation, in particular the ability to present a complete argument. The marking levels distinguished are (a) a complete, explicit argument, exhausting all possibilities, (b) partial attempts to explain or to illustrate by relevant examples, and (c) no relevant or correct comment. Most pupils who took the test showed signs of understanding the impossibility of obtaining 13p, but this is not possible to assess reliably from the scripts, these varied in the degree of explicitness of their arguments.

Add and Take tests the ability to recognise a general and explanatory argument as compared with one based on two cases. The question begins, Choose a number; add it to ten; take the original number from ten; add the two results. Two cases are to be produced;

then the pupil has to choose the better of these two arguments and explain the reason for his choice:

JANE says,

'Begin with 1, answer is 20.

Begin with 9, answer is 20.

So begin with any number between

1 and 9, answer will be 20.

The answer is always 20'.

BRENDA says,

'If n is the number you choose,

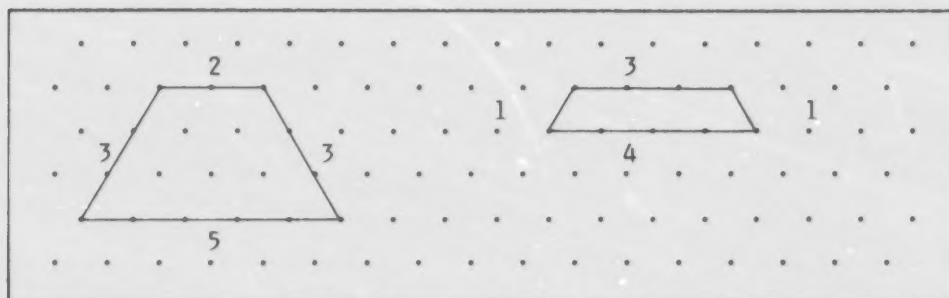
you have $10+n$, and you add $10-n$.

You add and take the same number

so you will always be left with

2 tens. The answer is always 20'.

Roofs offered a coding for roof-shapes drawn on an isometric dot grid, as shown:



Some possible and impossible roofs were then to be drawn from their given codings (2242, 3251 ...), then there was space for chosen trials, and the request to find and explain rules for deciding whether or not a given code will produce a roof. These rules, ' $b=d$ ', and ' $a+b=c$ ' may be expressed either verbally or algebraically; and a suitable explanation requires an argument based on the grid.

In this paper I have tried to show several examples of recent developments in assessment techniques drawn from various contexts, large and small scale, and covering the different purposes of evaluation, for persuasion, prediction and diagnosis.

Reference: SUMNER, R.: 1975, Tests of Attainment in Mathematics in Schools.
Windsor: NFER.

Dr Bell preceded the introduction to his paper by outlining the work of the Shell Centre for Mathematical Education at the University of Nottingham, one of the two major centres for research into mathematics education in Britain.

Discussion ranged over the varying purposes of assessment in mathematics education, and the corresponding styles and modes of questioning children. It is not possible to say whether there is close correlation between the results of traditional and more open-ended, investigatory tasks, since the purposes are somewhat different.

The importance of the APU type of investigation is that careful diagnosis of errors can lead to greater understanding of pupils' thinking. The APU work is the first major attempt in Britain to use oral techniques for such diagnosis in a systematic way. The English participants understood that verbal questioning plays a much larger role in the Soviet system, but time prevented our being able to pursue this issue as far as we would have liked.

EXAMINATIONS AND THE CURRICULUM

J Hersec, Executive Director, SMP

1. With no nationally laid down curriculum, the chief influences on what is taught in secondary schools are the examinations which pupils take at age 16 and age 18.

At age 16 most pupils (about 85% for mathematics) take an **external** examination (that is, an examination administered by an examining board, not by the school itself). The proportion of 18-year-olds who take a mathematics examination is about 6%. The different examining boards offer different syllabuses, but there is a substantial amount of common content between the various syllabuses.

For the examinations at age 18, recent discussions have led to agreement on a 'core syllabus', which is about half of the total syllabus. This core will be common to all syllabuses at this level. Between any two syllabuses there will also be further common content.

2. All examinations are now 'single subject'. Formerly, to obtain a certificate a pupil had to achieve specified grades in a number of subjects, some of which were compulsory. Now he receives a certificate showing his performance on any subject for which he has entered and achieved a grade, even if he has entered for only one subject.

3. Monitoring of standards Both the content of syllabuses, the examination papers set and the standards applied in marking the scripts of the examinations at age 18 are monitored by a central body, the Schools Council, but at present there is no similar check made on examinations at age 16. However, the Secretary of State for Education and Science has recently announced that the Schools Council is to be replaced by two bodies, one of which will be responsible for monitoring examinations at both ages.

4. Style of examinations Almost all of the external examining at age 16 and at age 18 is by timed, written, unseen papers, set by a panel of examiners appointed by the examining board to which the school is linked. No books are allowed in the examination room, although in some mathematics examinations the use of a sheet of formulae is permitted. The examiners who set the papers and mark the candidates' scripts are usually teachers in schools, but a number of teachers in colleges and universities are also involved, particularly for the age 18 examinations.

Although some teacher assessment is used in some examinations at age 16, there is none at age 18. In a few examinations (known as Mode 2 or Mode 3)* the teachers in a school have a considerable influence on the questions set in the written papers. The Cockcroft Report has recommended that more use should be made of assessment by the pupil's teacher.

However, a number of different styles of questions are used in written papers. Besides questions which are a straightforward test of a piece of technique, and those like text-book problems, multiple choice and short answer questions are also used, partly for administrative reasons and partly for mathematical purposes.

Short answer questions, for which the candidate writes his answer on a prepared answer sheet, can be checked easily and quickly. With large numbers of candidates several people will be involved in the marking. It is easier to obtain consistent standards where only the answers to questions are shown, to be marked 'correct' or 'wrong'. Multiple choice questions, for which the candidate selects the appropriate letter on the answer sheet, corresponding to the answer he believes to be correct, can be marked by computer or by unskilled clerical staff.

Short questions have been used (for example, in SMP O-level) to ensure that the whole of the syllabus is tested. Multiple choice questions have been used to examine qualities such as reasoning and understanding, rather than routine competence (for example, the University of London Board's A-level Syllabus D).

If any part of the examination is marked by the candidate's teacher, questions of fairness and establishing a consistent standard between different teachers arise, and this is a major difficulty in having a 'coursework' component. One attempt to deal with this problem can be seen in the University of London Board's **Pure Mathematics with Statistics**. The course consists of three statistical projects which the candidate takes into the examination and uses as a basis for his answers to the questions on the paper.

***Mode 1** examinations are those in which the syllabus and papers are prepared by the examining board and the candidates' scripts are marked by the examining board's examiners.

For **Mode 2** and **Mode 3** a school, or group of schools, or a Project prepares the syllabus. In **Mode 2** the papers are set and marked by the board's examiners. In **Mode 3** the papers are set and marked by the school. The examining board's task is to ensure that suitable standards are achieved for the various grades.

5. Results are given as **grades**. In examinations at 16 the terms 'pass' and 'fail' are not used, although there is an 'unclassified' grade for a very weak performance.

6. Examinations at age 18

It is convenient to begin with these examinations because the arrangements are simpler than for examinations at age 16. The title of the examination is **The General Certificate of Education, Advanced Level**, abbreviated to GCE A-level.

(a) There are seven GCE examining boards in England and one in Wales. With one exception, the Associated Examining Board, each is linked with one or more universities (see Appendix A). Although each board may be used chiefly by schools in its neighbourhood, a school can enter for the examinations set by any of the boards. Each board publishes a number of syllabuses in mathematics and sets papers on them. In addition, at this level, there are two Project examinations; the School Mathematics Project (SMP) and the Mathematics in Education and Industry (MEI), both administered by the Oxford and Cambridge Board. A school chooses which examination it will use.

In all cases the papers, set by the examiners appointed by the board, are sent to the schools and the candidates' scripts are then sent to the examiners who mark them.

(b) **Content of syllabuses**

(i) 'Single mathematics' (for pupils who study mathematics as only one of their academic subjects). All syllabuses are about half 'pure' mathematics (mainly the 'core' of paragraph 1) and the rest 'applied' mathematics. The applied part may be Newtonian mechanics only, or probability and statistics only, or some of each of these two subjects.

Usually there are two papers taken by each candidate.

(ii) 'Double mathematics' (for pupils who study mathematics as two of their academic subjects). Again half of the syllabus will be pure and the remainder applied. However, two patterns, each of four papers, are used. In one pattern there are two Pure Mathematics papers and two Applied Mathematics papers. In the other, the two papers set for single mathematics are used together with two others, known as 'Further Mathematics'.

(c) Apart from the obvious differences between syllabuses and papers resulting from the different content of the applied mathematics, there are other differences in content and style of questions, resulting from the 'modern mathematics' movement of the 1960s. While some syllabuses and papers remain more or less as they were in 1960, others, notably the SMP, differ from them substantially.

(d) **Grades** Results are given in terms of grades. Grade boundaries, in terms of marks, are norm-referenced (that is, it is assumed that the same proportion of candidates should be awarded each grade in successive years) although some adjustments, giving an element of criterion referencing, are made each year. Pass grades A, B, C, D, E are used; an O grade is considered the equivalent of a pass (grade A, B, C - see paragraph 7) at Ordinary Level, and f is used for fail. The proportion of candidates to be awarded each grade, expressed as a percentage of those who take the mathematics examination at this level, is as follows:

A 10%; B 15%; C 10%; D 15%; E 20%; O 20%; f 10%.

(Examples of syllabuses and papers for GCE A-level were available at the Seminar).

7. Examinations at age 16

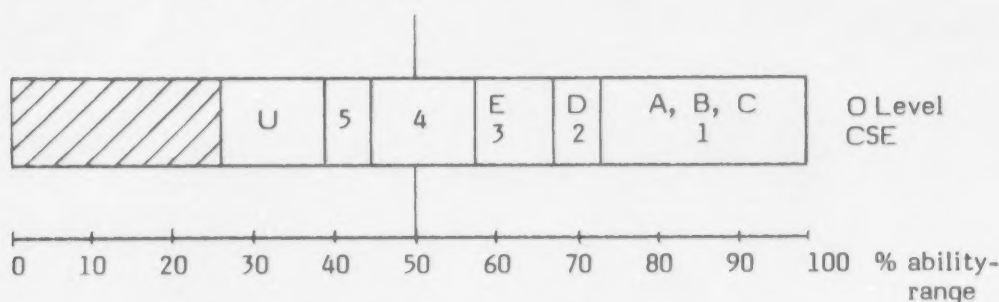
At present there are two systems in use, but the government has declared its intention to combine them in a single system, if possible, during the 1980's. The Secretary of State for Education and Science has asked the examining boards to prepare statements of criteria which he will approve and which all examinations under the new single system must satisfy. The criteria for mathematics will include a core of content which must be contained in any syllabus.

(a) All the GCE examining boards listed in Appendix A offer **General Certificate of Education Ordinary Level (GCE O-level)** examinations. These are intended for roughly the most able 25% of pupils.

(b) In addition, there are thirteen examining boards for the **Certificate of Secondary Education (CSE)** (see Appendix B). These boards operate on a regional basis. The CSE examination was intended to cater for pupils whose abilities lie within the 25th to 60th percentiles. In practice the highest grade awarded on a CSE certificate is linked with the top three grades on GCE O-level, so that CSE examinations are taken by some pupils of high ability. In practice, pupils whose abilities are much less than the intended lower level are also entered for mathematics in CSE.

(c) Grades

Performance in the O-level examination is expressed in terms of grades A, B, C, D, E and U - 'unclassified'. Grades A, B and C are normally considered to be grades of 'pass'. For the CSE, grades 1, 2, 3, 4, 5 and U are used. Grade 4 CSE should represent the performance of the average pupil, taking the whole of the age-group into consideration; other grades, and the relation between grades for the two examinations are shown in the diagram below.



(Examples of GCE O-level and CSE syllabuses and papers were available at the Seminar).

8. National monitoring of the curriculum

In 1975 the **Assessment of Performance Unit (APU)** was set up within the Department of Education and Science to provide information about general levels of performance of school pupils and how these change over the years. "The aim of the APU is to produce and make generally available overall national pictures of certain aspects of actual pupil performance. It is not concerned to assess the performance of individual children, individual schools, or individual local education authorities, nor to make judgements about or directly influence the curriculum or teaching methods within any particular school."

In mathematics a two-way categorisation, by content and by learning outcome, is used. The five main content areas are - number, measures, algebra, geometry, probability and statistics; the five learning outcomes are - concepts, skills, applications, problem solving and investigations, attitudes. The test items used have attempted to cover the full variety of mathematics to be found in different schools.

Printed tests requiring short written answers are given to samples of pupils at ages 11 and 15. About 14 000 pupils are tested in each sample, that is about $1\frac{1}{2}$ - 2% of the age-group. In addition a practical test is used and an attitude questionnaire is used.

The early surveys concentrated on the testing of skills and concepts, but later tests included tests of the other learning outcomes. Summary reports will be produced after the first five years of testing, by which time it is hoped that sufficient data will be available to show whether trends can be discerned.

(Copies of abbreviated APU reports were available at the Seminar).

9. Changing the curriculum

Because the A-level examination has come to play an important part in selection for university education, the A-level syllabuses are strongly influenced by the universities. Indeed, the 'core syllabus' mentioned in paragraph 1 is the result of university pressure. There is no doubt that the introduction of this core syllabus will affect both content and style of teaching in A-level courses. If a change in A-level seems desirable, but appears to lead to students being less well prepared for university courses, it will be difficult to introduce.

The starting point for an A-level course is O-level. Any change proposed at O-level must not weaken the basis that the course provides for A-level. Thus, in turn, it is difficult to make changes at O-level in mathematics, even though, for many who take O-level mathematics, it may be the end of their mathematical studies.

Because of the link between CSE grade 1 and O-level, CSE syllabuses are strongly influenced by O-level syllabuses, so that here, too, change is not easy.

Pressure for changes comes from four sources.

- (a) Teachers in schools may suggest changes by writing to the examining board.
- (b) Each examining board has its own subject panel for mathematics which reviews syllabuses and papers and suggests changes.
- (c) Major Projects (SMP, MEI) have proposed and introduced some of the most radical changes.
- (d) A number of Mode 2 and Mode 3 schemes, especially for CSE, have been accepted, which differ markedly from the CSE board's own scheme. If successful they may be made widely available, and may produce changes in the board's scheme.

Recently the 'Cockcroft Report' has criticised CSE syllabuses and made recommendations which also affect O-level and A-level syllabuses and examinations. It will be interesting to see how far these recommendations are followed up.

APPENDIX A

Examining Boards for the General Certificate of Education (GCE)

Associated Examining Board
 Cambridge University Local Examinations Syndicate
 Joint Matriculation Board (**Northern Universities**)
 Oxford Delegacy of Local Examinations
 Oxford and Cambridge Schools Examination Board
 Southern Universities' Joint Board
 University of London School Examinations Department
 Welsh Joint Education Committee

APPENDIX B

Examining Boards for the Certificate of Secondary Education (CSE)

Associated Lancashire Schools Examining Board
 East Anglian Examinations Board
 East Midland Regional Examinations Board
 London Regional Examining Board
 North Regional Examinations Board
 North West Regional Examinations Board
 South-East Regional Examinations Board
 South Regional Examinations Board
 South-Western Examinations Board
 West Midlands Examinations Board
 West Yorkshire and Lindsey Regional Examining Board
 Yorkshire Regional Examinations Board
 Welsh Joint Education Committee

In introducing this factual paper, Mr Hersee emphasised that examinations exert a dominating effect on mathematics teaching in Britain. He wondered whether Soviet work on minimum standards of achievement for all might be a comparable situation.

British Primary Schools are free of external examinations, and it is surely no coincidence that the quality of teaching is probably highest at that level.

Mr Hersee also underlined the fact that the work of the Assessment of Performance Unit is essentially different from that of public examinations, since it monitors overall standards, rather than assesses individual pupils.

The discussion centred on the use of investigations and practical testing. There is difficulty in organising this kind of assessment procedure. Furthermore, realistic problems tend to use insubstantial mathematics; extended investigations using "real" mathematics may be possible at school level, for example in the work of the Mathematics Applicable Project as described in Mr Watson's paper, but they are not a suitable means of examining individual pupils.

Session 10. Training Mathematics Teachers in the USSR

REQUIREMENTS FOR THE TRAINING OF MATHEMATICS TEACHERS

Professor N E Shkil, Head of the Kiev State Pedagogical Institute

The scientific and technological revolution gave rise to far-reaching changes in the forces of production in our country. Science is increasingly penetrating all spheres of human activity and is becoming a part of the forces of production. Mathematics, which in the present day is necessary everywhere and to everyone, occupies a special place among the sciences. Without mathematical calculations it is impossible not only to construct a space ship, build a plane or a power station but also to erect an ordinary building. It is difficult to conceive of any branch of human activity in which it would be possible to get by without mathematics and its methods. On this subject Academician A I Berg said "We live in an age of the mathematisation of all the natural and humanitarian sciences". And therefore it is no accident that in the Programme of the Communist Party of the Soviet Union (CPSU) among the leading sciences whose development determines subsequent scientific and technological progress, mathematics occupies first place. This is why it is so important that everyone should have a knowledge of this science.

Only the secondary school can provide our growing young people, who will later go out into the world of human activity, with the necessary mathematical knowledge. And therefore it is not unnatural that as science in general and mathematics in particular develop, so the school mathematics course is modernised. This has happened in our country over the last one and a half decades. Some of the complex concepts of higher mathematics have entered school mathematics education alongside its traditional content which is usually referred to as elementary mathematics. Among these concepts are: set, function, limit, continuity, derivative, indefinite and definite integral, differential equations. This in its turn gave rise to new problems and new demands on the training of mathematics teachers.

The training of mathematics teachers, like the training of teachers of other subjects, must take as its starting point the aims of secondary education. These aims are formulated in the Programme adopted by the CPSU at the 22nd Congress: "Secondary education must provide a sound knowledge of the foundations of the sciences, a mastery of the principles of the communist world view, polytechnic training and training for work in accordance with the rising level of the development of science and technology, taking into account the needs of society and the abilities and wishes of the pupils, and the moral, aesthetic and physical education of the healthy growing generation" (Programme of the CPSU, 1974, p 123).

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In order to train a qualified teacher of mathematics it is necessary to know what that means, ie know the "Professional profile".

In the professional profile of the mathematics teacher it is possible to identify two aspects: 1, a general pedagogic quality; 2, a quality determined by the speciality, the subject being taught. The general pedagogic and special qualities always show themselves in the teacher's activity as being in unity with each other, interconnecting and interacting.

The general pedagogical qualities, we believe, include the following:

- a. curiosity;
- b. psychological readiness for pedagogical work;
- c. an influential pedagogical personality;
- d. an ability to maintain a close contact with pupils, parents and society;
- e. psychological-pedagogical powers of observation;
- f. pedagogical discourse;
- g. intellectual activity;
- h. the ability to take part in social work;
- i. organisational skills, particularly in the field of up-bringing;
- j. high moral qualities.

The formation of these qualities in students who are going to become teachers is first and foremost the task of the social and psychological-pedagogical disciplines and pure methodics. In the curricula of Soviet teacher training colleges up to 30% of academic time is devoted to the study of these disciplines.

The professional qualities of a mathematics teacher are defined by the demands placed upon mathematical inclinations and individual abilities.

Research by Soviet psychologists has shown that secondary school pupils can successfully master mathematical knowledge given average human abilities and a methodologically sound teacher. The Soviet psychologist V A Krutetsky holds that pupils' mastery of mathematical knowledge is conditioned by diligence in study, a well developed memory, attentiveness and powers of observation, and interest in mathematical knowledge. At the same time special mathematical qualities which the teacher of mathematics must possess also play an important role in the mastery of mathematics and the development of mathematical abilities. These qualities are the main component of the professional profile of the mathematics teacher. According to V A Krutetsky the most important of these qualities are: interest in mathematical knowledge; a mathematical turn of mind; the

ability to see the theoretical and practical significance of mathematical knowledge; a mathematical bent in the powers of observation which reveals itself in the easy and rapid discovery of mathematical errors and non sequiturs in mathematical judgements; spatial awareness and orientation in spatial constructions and projections; a memory for mathematical symbols, formulae and exercises; mathematical imagination; a formalised grasp of mathematical material; the ability to grasp the formal structure of a problem; logical thinking in the sphere of qualitative and spatial relations, of numerical and sign symbolisation; a tendency to think in terms of cause and effect; an ability to think in mathematical symbols; a bent towards analysis and synthesis in the sphere of mathematical phenomena; the ability to argue from the particular to the general and by analogy (inductive, deductive and analogical reasoning); flexibility in mathematical thinking; ability to grasp quickly and generalise mathematical relations and operations, the desire to search for unknown generalisations; an inclination to abbreviate the process of mathematical reasoning and systems of operation, the ability to think in abbreviated structures; speed in redirecting the thinking process and switching from a direct to an inverted train of thought; a desire for easy, simple and economical solutions.

The cited professional requirements of the mathematics teacher cannot be made absolute. In the process of mastering the theoretical knowledge and especially during teaching practice (in college and working in schools) they are perfected. However, even at the stage of selecting applicants to teacher training colleges for the special study of mathematics it is necessary to look for a certain level of these qualities among the school leavers. A test of mathematical thinking, powers of mathematical observation, spatial awareness, logical thinking (certainty, consistency and proof-based judgement), the applicants' inclination towards a pedagogical expression of his mathematical knowledge, must be included in the examination or interview for those seeking a place in the faculties which train teachers of mathematics. In addition it is essential that the curriculum should foresee the development of these qualities both in the pedagogical and in the special mathematical disciplines, but particularly in the students' practical work in schools, beginning with the first year (while continuing study or by interrupting it). It is also expedient to include in the curriculum special courses on psychology and pedagogics in which the professional qualities of the mathematics teacher may be examined.

Along with pedagogical and professional inclinations the mathematics teacher must have a profound knowledge of the fundamental mathematical sciences. These include the mathematical disciplines of analysis, geometry, algebra and theory of number. It is in these disciplines that the student becomes acquainted with the fundamental concepts of the mathematics which is studied in secondary schools. The teacher's level of

mathematical culture wholly and completely depends on the extent to which these disciplines are mastered in college. If future teachers study these disciplines in all their aspects this will enable them to find their bearings in mathematics and retain a store of knowledge which extends far beyond the bounds of the school syllabus. This will enable the future teacher to have a clear, scientific conception of the up-to-date content of the fundamental mathematical concepts such as number, function, limit, derivative, definite and indefinite integral, vector, length of a line, area of a plane figure, area of a surface, volume of a solid, length of an arc, etc. Thus he will have not only perfect control over his special subject but also a deep understanding of the fundamental ideas of mathematics teaching reflected in the new syllabuses and textbooks, a command of modern effective teaching methods, and a knowledge in outline of the achievements of modern mathematical science and the history of its development. And, finally, a study of the above-mentioned disciplines will teach future teachers the ability to think and reason with rigour and precision.

The fundamental link in the professional training of mathematics teachers is the system of methodological training which equips students with the theory behind the teaching of their subject and with practical abilities and skills in problem-solving and working in schools. The basic components of this system - a course of lectures, methodological seminars, practical laboratory sessions, teaching practice, course-work and dissertation and practice in problem-solving - provide the future teacher with the necessary knowledge and skills for school mathematics teaching.

In the system of methodological training of mathematics teachers particular attention should be paid to the organisation of students' research work. Research is one of the basic functions of a mathematics teacher's professional activity. It is closely bound up with the organisational and educational functions of the teacher's pedagogical activity.

Research work can be organised in a wide variety of ways. The most effective of these are: introducing students to problems which reflect the scholarly interests of the department; getting them to produce dissertations and course work and participate in special seminars; and encouraging them as much as possible to undertake the independent study of specific topics in the syllabus making extensive use of supplementary literature.

At the same time the system of training of mathematics teachers in teacher training colleges is still in need of further improvement.

Although the existing curriculum does on the whole provide a high level of mathematical, psychological-pedagogical and methodological training for students who are going to

become mathematics teachers, nevertheless it needs to be worked out more completely in the following ways:

1. As a result of the fact that the course of elementary mathematics has been excluded from the curriculum of teacher-training colleges and also because the number of hours spent on the methodics of mathematics teaching has been reduced, the level of professional and, more particularly, of methodological training for mathematics teachers has dropped slightly. The attempt to distribute the syllabus topics of the elementary mathematics course among the basic mathematics courses (algebra, geometry, mathematical analysis) and the introduction of a course on the modern foundations of school mathematics have not proved justified. The basic mathematics courses have their own aims and are in fact taught traditionally, but the course on "the modern foundations of school mathematics" turned out to be remote from the content of school mathematics. Therefore it is necessary to introduce into the curriculum a course of elementary mathematics which will act as an intermediate link between college and school mathematics.
2. Optional mathematics courses aiming at a differentiated approach to school teaching have been introduced into secondary schools. However, students in teacher training colleges are not sufficiently trained to run optional courses. Therefore it has become necessary to work out how options should be organised in schools (from the point of view both of content and method), and to give future teachers a thorough acquaintance with these matters.
3. Given the increasing importance of computers in all sectors of the country's economy, teacher training colleges ought to introduce at least a short course in applied mathematics which would include programming for computers and micro-computers, elements of probability theory and mathematical statistics, computer mathematics and optimisation methods.
4. The mathematics teacher must have a good knowledge of school textbooks so that he can not only solve the problems and do the exercises contained in them but also master their theoretical foundations and methodological conceptions. Obviously, curricula must make provision for the possibility of a special study of school textbooks. This is all the more necessary given the introduction at the present time of new textbooks into secondary general schools.

The improvement of the training of students of mathematics in teacher training colleges in the USSR along the lines indicated will make it possible to raise this training to a new

and higher level, thereby successfully solving the problems which the 26th Congress of the CPSU set for Soviet higher pedagogical colleges and secondary general schools.

The following additional points arose in discussion.

In the Ukraine (for example) they have been strengthening the ties between colleges and schools, particularly studying the practice of the best teachers. Lecturers and professors have assisted with optional classes in the schools. They have also experimented with calculators in school and the results are reported in "Mathematics at School" No.1, 1981. As a result of the experiment the Ukraine authorities have approved the use of calculators in schools. This is one of the regions which has also been experimenting with the new geometry text.

Concerning the problem of selecting students with the desirable qualities listed in the paper, Professor Shkil said that some school pupils were identified as being suitable for teaching, and grooming began while they were in the 9th and 10th grades at school. In Kiev two pupils from each school (around 290 schools) provided practically the whole intake for the Faculty there.

The research in which students were engaged took place in "problem groups" associated with each chair in the Faculty, and was undertaken during years 3, 4 and 5 of the course, being led by the Professor. Some students would go on to postgraduate research into pedagogical problems.

Professor Shkil agreed that the study of elementary mathematics from a higher point of view was important and was being built into courses in mathematics.

The Kiev Institute has a Faculty for the retraining of School Principals, and every Director and Assistant Director of a school is required to attend a 2 months course every 5 years. Other Institutes for the advanced training of teachers extend such provision to all teachers. Each Institute has two experimental schools, one urban and one rural. Teachers who have been on in-service courses are observed and feed-back obtained for future courses.

THE PEDAGOGIC AND METHODOLOGICAL TRAINING OF TEACHERS OF MATHEMATICS

Professor R S Cherkasov, Moscow State Pedagogical Institute; Editor-in-Chief,
"Mathematics at School"

The pedagogic and methodological training of teachers of mathematics in teacher training colleges is carried out by means of a set of academic disciplines and various forms of teaching practice. In all, the time spent in class on pedagogic and methodological disciplines amounts to 20% of total academic time. However if one takes into account the time spent on teaching practice, this figure rises to 30%.

The pedagogic disciplines include:

1. developmental physiology;
2. pedagogics (including history of pedagogics);
3. psychology;
4. introduction to special subject;
5. special courses and seminars in accordance with choice.

The methodological disciplines include:

1. basic concepts of the school mathematics course;
2. methodics of teaching mathematics;
3. practice in problem-solving;
4. special courses and seminars in accordance with choice;
5. practice in measuring and calculating.

Among the optional disciplines are history of mathematics, the methodics of education, careers for pupils and other subjects recommended by the respective departments. For each of the disciplines studied the students take an examination on the theoretical part and a test on the practical part.

In addition the students do course work in pedagogics and the methodics of mathematics teaching. In his or her course work the student must demonstrate both knowledge of the theory behind the question chosen and the ability to use the experience acquired on teaching practice in dealing with the subject.

Teaching practice includes:

1. Practice in children's summer camps (4-8 weeks).
2. Educational and methodological practice in schools (6 weeks)
3. Probationary practice in schools (7 weeks)

On the basis of the results of their teaching practice students are assessed and given one of the following grades: unsatisfactory; satisfactory; good; excellent. Students who are graded as unsatisfactory on the basis of their teaching practice may be excluded from the college.

On completion of the course of pedagogics students are expected to have acquired knowledge of the following:

1. The foundations of pedagogics (the subject and method of pedagogics, the development and education of the individual, the goals and problems of education, the system of education in the USSR).
2. The theory and practice of education (the essence and principles of education, general methods of education, the education of the individual within the collective, the formation of a world-view, moral education, education for work, aesthetic education, physical education, social organisation in the school, extra-curricula and extra-mural work, links between school, family and society).
3. Theory of education and teaching (essence of the teaching process, principles of teaching, content of education, teaching methods, the organisation of lessons, the testing and assessment of pupils' knowledge, abilities and skills, the roles of the teacher and the form monitor in solving educational and teaching problems, the administration and running of the school.)

From studying the methodics of mathematics teaching students must acquire knowledge of the following:

1. The goals of mathematics teaching in secondary school. The significance of the school mathematics course in relation to education as a whole.
2. The content of school mathematics education and the direction of its development. Facilities for teaching mathematics.

3. An analysis of textbooks and handbooks on mathematics.
4. The implementation of the principles of Soviet didactics in mathematics teaching.
5. Mathematical concepts, propositions and proofs.
6. The methods of teaching mathematics.
7. The role of problems in mathematics teaching.
8. The nature of a mathematics lesson.
9. The special nature of the work of teachers in schools and classes where mathematics is studied in greater depth.
10. Optional courses in mathematics (aims, content and how to conduct them).
11. Extra-curricula work in mathematics, mathematical olympiads.
12. The methodics of the study of arithmetic and elementary algebra in classes 4-5; of the courses in algebra and geometry in classes 6-8 and 9-10; of basic mathematical analysis in classes 9-10, and of computer mathematics and programming.

Students do their teaching practice in school in their penultimate and final years.

The aims of teaching practice are the following:

- a. to deepen and reinforce the theoretical knowledge which the students have already acquired and to teach them to apply this knowledge in practice;
- b. to develop the students' ability to observe and analyse the work of the educator;
- c. to teach the students to work with children taking into account special features relating to their age and individuality;
- d. to teach students to prepare and conduct mathematics lessons independently using various methods and to encourage pupils' active involvement in the subject;

- e. to prepare students to perform the duties of the form monitor, to work within the pupils' collective and with social organisations, and to carry out individual educational work with pupils;
- f. to develop and reinforce students' love for the teaching profession, to stimulate their desire to study special and pedagogical disciplines and to perfect their pedagogical abilities with a view to preparing themselves for the creative solution of educational problems;
- g. to inculcate the skills of caring for the health of school-children;
- h. to inculcate an interest in academic and research work in the sphere of the pedagogical sciences, in contacts with and analysis of the pedagogical experience of teachers;
- i. to encourage students to engage in socio-political work and pedagogical propaganda among the population.

Students complete their course at teacher training college with state examinations in philosophy, mathematics and pedagogics with the methodics of mathematics teaching.

The state examination in pedagogics with the methodics of mathematics teaching may be replaced by the defence of a dissertation on pedagogics or the methodics of mathematics teaching. The defence of a dissertation is allowed in the case of those students who make good progress and show an inclination towards research into questions of education, didactics or the methodics of mathematics teaching. As a rule the dissertation is a continuation of coursework already completed by the student.

The increasing role of the school in the life of modern society means that the pedagogic and methodological training of future teachers must face up to a series of new and complex problems requiring a higher level of theoretical and practical training for future students.

At the present time teacher training colleges are introducing a system whereby students carry out academic research into all the subjects on the curriculum, including the various pedagogical and methodological disciplines. The academic work serves as a basis for the broad development of student research carried out in students' societies. It is intended that in the near future continuous teaching practice in schools, starting from students' first year, should be introduced.

In a number of teacher training colleges such practices have already been tried out and have produced good results. Considerable experimental work is being carried out to investigate the effectiveness of methods of orienting pupils in the senior classes of secondary schools towards the teaching profession.

The mathematics teacher's professional training requirements are being worked out - requirements which include ideological-theoretical training, essential to the future teacher qua educator and didactician; pedagogical training, essential to active pedagogical work, academic training which defines the individuality of the teacher qua specialist in his or her subject; and methodological training, which enables the teacher to make creative use of the knowledge and skills he has acquired in the social, mathematical, pedagogic and methodological disciplines in his work in education.

The training of mathematics teachers for secondary schools is carried out in universities as well as in teacher training colleges. In universities the volume of work in the pedagogic and methodological disciplines is less. But the university curriculum provides a deeper academic training for students of mathematics and all students are required to complete a dissertation.

In his presentation, Professor Cherkasov added the following points to his paper:

1. 80% of the specialist mathematics teachers for grades 4 to 10 are trained in the Pedagogical Institutes. The remaining 20% come from Universities. About 15% of those who enter Institutes are eventually rejected, because they do not have the desired qualities.
2. The pedagogic disciplines precede study of the theory of teaching mathematics.
3. Institutes draw on the experience of other countries. The practice of starting school at age 5 in Britain has been used to support moves towards lowering the age in the USSR.
4. Students are encouraged to criticise the results of their teaching and not to panic into developing negative attitudes towards pupils who make mistakes.
5. To further smooth the transition from "student" to "teacher", teaching practice now begins during the second half of the first year and continues until the

end of the 5 years' training. Some problems, such as how to encourage individual development in a large class of pupils, can only be solved by practical experience in the classroom; it cannot be taught to student-teachers. Students conduct experimental teaching at school, take part in student Olympiads and in competitions for the best research papers. They also help with pupil Olympiads through correspondence in the pages of "Kvant".

Professor Cherkasov concluded by saying that results were satisfactory but this was not enough; they ought to be good!

In response to questions, further points emerged:

1. When teachers are posted to schools they normally begin teaching the lower grades (from 4 upwards) and would not teach grades 9 and 10 at first; the school decides.
2. Details of how students are gradually introduced to teaching would require another symposium. Students are encouraged to be responsible for their own development.
3. There were separate Institutes for training teachers for grades 1 to 3, and these teachers taught a range of subjects. It was suggested by a British delegate that another symposium on the training of Primary teachers could be of value.

THE MATHEMATICAL TRAINING OF MATHEMATICS TEACHERS

Dr N J Vilenkin, Moscow State Correspondence Pedagogical Institute

1. The aims and content of mathematics training for mathematics teachers

Mathematics training in higher educational institutions may be divided on the basis of its aims into three types:

- a. university (in mathematics faculties);
- b. technical (also training in economics institutes and in physics and physico-technological faculties in universities);
- c. pedagogical.

The differences in the aims of these three forms of mathematical training are as follows:

- a. The main purpose of the universities is to train specialists for research work. Therefore the teaching of mathematics in universities must develop students' skills in this direction, for which purpose it is necessary to conduct the teaching at a high level of rigour and to acquaint students with the latest achievements of mathematical science.
- b. In technical institutes and those university faculties which resemble them (in this connection) mathematics is not an end in itself but an instrument which is essential for the mastery of technical skills and knowledge. Therefore the greatest attention must be devoted to the applied aspects of mathematics and rigorous exposition is less important than the use of visual models, appeals to intuition and common sense, and a clear emphasis on the algorithmic and computational aspects of the material.
- c. The aims of education in the special mathematics courses in teacher training colleges are different from those set out above insofar as the future teacher of mathematics is hardly likely to have to carry out research within the field of the mathematical sciences or concern himself with the solution of applied mathematical problems. At the same time his mathematical training must be sufficiently broad for him to be able to carry out his teaching not only in



circumstances where syllabuses and textbooks are "stable" but also at times when various syllabus changes are taking place and when one textbook is being replaced by another which is more appropriate in the modern context. Mathematical training of this sort must enable the teacher to make an independent assessment of the quality of various approaches to the teaching of mathematics in schools, to understand both the global aims of teaching and the nuances in the presentation of individual topics. The teacher must be able to teach both at a high level of rigour and by means of visual models, depending on the stage reached by his pupils. He must also understand the relationships between the concepts he teaches, have mastered the axiomatic method, be able to solve difficult problems, be capable of taking optional courses in school, and be able to grasp and properly assess original solutions to problems which more gifted children may produce, without simply insisting that they use the standard method.

Great demands are being made upon the mathematical training of mathematics teachers as a result of the fact that computer technology is now beginning to penetrate schools. The teacher must be acquainted with programming, without which he will not be properly prepared for the age when widespread use will be made in schools of the various possibilities of links with computers.

It is of the greatest importance that the mathematics teacher should be acquainted with the history of his subject, since experience shows that many of the psychological difficulties that pupils experience as they study the school mathematics course reflect the difficulties which arose in the course of the development of the science (difficulties with operations using fractions, negative and complex numbers; mastery of the axiomatic method; understanding the essence of mathematical analysis and so on). A knowledge of the mathematical development of the science will also enable the teacher to understand the various approaches to the basis of the school geometry course (that founded on the axioms of Euclid and their development by D Hilbert, the vector approach on the basis of the axioms of G Weil; and the metric approach in various forms, eg that of A A Kolmogorov and A V Pogorelov).

2. The mathematical training of mathematics teachers in the USSR

The points which have been made above concerning the purposes of the mathematical training provided by mathematical faculties of teacher training colleges must be taken into account when the curricula and syllabuses of these institutions are being drawn up. The system of training which exists in the USSR is basically directed towards the achievement of all the afore-mentioned aims.

Of a total number of 3705 hours devoted to speciality No. 2104 - mathematics - 2395 hours are set aside for the study of mathematical disciplines (the remaining hours are taken up by courses in physics and astronomy, socio-political disciplines, and also pedagogical disciplines and a course in the methodics of mathematics teaching). The basic core of the mathematical training consists of three courses: "Mathematical analysis and the theory of functions", "Algebra and theory of numbers" and "Geometry" which, with the inclusion of exercises, occupy 580, 370 and 400 hours respectively. In addition there are other mathematical courses such as "Number systems" (55 hours), "Mathematical logic" (20 hours), "Probability theory" (60 hours), "Computer mathematics and programming" (115 hours) and "The modern foundations of the school mathematics course" (70 hours). 175 hours are devoted to "Practice in problem-solving" which deals with more difficult problems from school mathematics.

As a result of the introduction into schools of elements of differential and integral calculus the pedagogical significance of the course in mathematical analysis has increased. Now this course consists of the following sections: introduction to analysis; differential and integral calculus for functions with one variable; series (numerical and functional); theory of metric spaces; differential and integral calculus for functions with many variables; ordinary differential equations; set theory and the Lebesgue integral; the Fourier series and the theory of analytic functions. The material is presented with the classical rigour which had come into being by the end of the 19th century (the language of " ϵ, δ ", of limits as the basis of mathematical analysis etc). Theory of logic symbolization is fairly widely used and much is expected in the way of ability to solve problems of classical analysis.

The courses on "Algebra and the theory of numbers" and "Geometry" are equally fundamental. In the former, attention is devoted largely to the study of the basic algebraic structure - groups, rings and fields and linear spaces over fields. Questions of classical advanced algebra are given slightly less coverage than previously, and the exposition of elements of the theory of numbers is subordinated to general algebraic ideas. This course is characterised by its considerable orientation towards set-theory; apart from the language of mathematical logic, wide use is made of binary relations and their properties, and basic concepts are defined on the basis of the "breaking down" of corresponding sets into classes of equivalence etc.

The Geometry course deals with affine and projective geometry of multi-dimensional space, the general theory of second order shapes, methods of representing geometrical figures, the theory of polyhedra, the axiomatics of geometry (Weil and Hilbert), elements

of differential geometry and topology. The syllabus for this course and the textbooks based on it have been strongly influenced by "Bourbakist" views which received a lot of publicity at the end of the 60's and beginning of the 70's, at the time when these syllabuses were being adopted and the textbooks compiled.

Apart from the relevant section in the geometry course, other courses which are devoted to the foundations of mathematics are "Mathematical Logic" and "Number systems". The first of these covers the calculus of statements and predicates, the theory of algorithms and Godel's completeness theorem. The second deals with the axiomatic construction of basic number systems (of natural numbers, whole numbers, rational numbers, real numbers, complex numbers). The exposition is extremely rigorous and includes the ideas of formal and polyformal theories; an examination of problems of the non-contradictoriness of the arithmetic of natural numbers, the theory of ordered sets and algebraic systems and algebras, the theory of the expansion of algebraic systems and algebras, the theory of fixed fields and also a study of quaternions and the Frobenius theorem.

The mathematical training in the other mathematical disciplines is of a similar nature. The course entitled "The modern foundations of the school mathematics course" occupies a special position in that its aim is to establish links between the mathematical courses outlined above and the course of the methodics of mathematics teaching. In this course the most important question relating to school mathematics are examined "from above". In addition, students attend special courses on mathematics and participate in special seminars.

3. Methods of improving the mathematics training of mathematics teachers

The existing system of mathematical training for mathematics teachers came into being in the late 60's and early 70's, when an attempt was made to reorganise the entire school mathematics course on the basis of set theory. This also explains the Bourbakist tendency of most courses, the extra emphasis on questions related to set theory, mathematical logic, general algebra and so on. Experience has shown that this type of presentation of mathematics in the first place is too difficult for the majority of students and in the second place creates a misleading impression of mathematics as a formal science removed from applications. At the same time, the reorganisation of school mathematics education which is taking place at present, and the aim of which is to emphasise the applied aspect of the course and bring it into line with the needs and requirements of the vast majority of school leavers, makes many of the above mentioned questions remote for the future teacher of mathematics. We note that a rejection of the Bourbakist tendency in the teaching of mathematics in schools, the increased emphasis on applied aspects, the growing importance of questions relating to computer mathematics and so on are phenomena which can be observed throughout the world and are the inevitable result of

the introduction of cybernetics into industry and the appearance of computers in all spheres of human activity.

On the strength of all that has been said above, it is becoming essential to bring the training colleges' mathematics courses into line with the new tendencies, to reject the too abstract presentation of many topics and to increase the importance of those subjects which have a direct contact with the school mathematics course. The future teacher hardly needs to study such subjects as the Lebesgue integral and theory of measure, or to go into the details of the properties of geometric shapes and linear operators and multi-dimensional space, the properties of topological space, differential forms and so on. On the other hand it is necessary to study in greater depth the properties of elementary functions and their relationship to basic number structures (the additive group of real numbers, the multiplicative group of positive numbers and the group of complex numbers which are equal in accordance with the modulus 1). It is necessary to investigate the general principles of the symmetrization of algebraic structures and the use of these general principles in the construction of number systems. Particular attention should be given to a range of questions connected with the measurement of geometric magnitude, the properties of plane and solid figures, the various definitions of the length of an arc and the area of a surface and the links between these definitions.

At the present time a range of topics relating to the foundations of mathematics are being studied in several different courses (geometry, number systems, mathematical logic), which inevitably leads to unnecessary repetition. It has become vital to develop a single course on the foundations of mathematics which would cover all of these topics.

In the majority of courses which are run in teacher training colleges it is essential to emphasise the applied aspect. For example it would be extremely useful to show how differential equations of mathematical physics are solved, to talk about the application of differential geometry to the solution of problems relating to cartography and machine parts, to point out the applied aspects of algebra, to deal with the use of probability theory for the solution of practical problems, and so on. Only as a result of this sort of approach will mathematics appear to the future teacher not as a formal schema in which, as B Russell said, it is not clear what is being talked about, but as a powerful tool for acquiring knowledge of and transforming nature.

It is essential to show - far more than is the case at present - not so much the technical methods for solving individual problems but rather the great ideas which lie, for example, at the root of mathematical analysis - the idea of the local linealisation of functions,

leading to the various sections of differential calculus, and the idea of the approximation of a given function by step functions, leading to the various definitions of the integral.

At the same time it would be very useful to devote some time during the mathematics course in teacher training colleges to individual and narrower topics connected with school mathematics. For example, it would be worth talking about various methods of calculating the value of π , linking the beginnings of a study of Fourier series with transformations of the products of trigonometric functions into the sums of such functions, showing various methods of calculating the values of elementary functions, explaining the role of symmetric multinomials in the solution of equations and inequalities, and so on.

We believe that the modern reorganisation of the mathematics course for teacher training colleges in accordance with the above mentioned principles will lead to a subsequent improvement in the training of mathematics teachers, who, after all, must lead the teaching of mathematics into the 21st century.

The following points were discussed in question time.

1. In comparison to British practices, the time devoted to mathematics in the pedagogical institutes (= teachers' colleges) was extensive. Dr Vilenkin said it was sometimes challenged by the pedagogues.
2. Most mathematics students at universities go on to specialist Research Institutes studying industrial problems. As in Britain, a few go on to fundamental research. Less than 20% of future teachers are trained at universities, and they receive no pedagogical training; the Soviet Union has no equivalent of the PGCE. The problem is that it is not known in advance which students will go into teaching.
3. The university courses have the following durations:
 - 4 years for mathematics degree
 - 5 years for mathematics-with-physics degree
 - 5 years for physics-with-mathematics degree.
4. All three papers on teacher training had aroused great interest among the British delegation, several of whom work in this area. They would welcome the opportunity to present papers at a symposium on teacher training in Britain and to benefit by comments and questions from their Soviet colleagues. For example, reference was made to "The Mathematics Teacher Education Project" (MTEP, Blackie, 1980), a collaborative exercise which produced materials appropriate for use in the one-year teacher training courses for university graduates in Britain. Classroom research work was also mentioned briefly.

CLOSING DISCUSSION

The final session of the seminar was split into two parts. The first was an open forum in which all participants were invited to give their reactions to the Seminar and its value, and to put forward their ideas for follow-up activities. Many participants accepted this invitation. In the second part, Dr Fletcher and Professor Monakhov expressed their own views both on the Seminar itself and on future collaboration in mathematics education between our two countries.

There was widespread agreement on all the major points raised. Participants from both countries had found the Seminar of considerable value. Much had been learned about mathematics education in the other country, while the challenge to some of one's own professional assumptions stimulated fundamental thinking about the purposes of school mathematics, and about what is both desirable and possible in mathematical education. The Seminar had been enjoyed by everyone who took part, and cultural and language difficulties had not prevented the establishment of genuine friendships between the two delegations.

It was agreed that the phase of information-exchange was now over. Before another general seminar was appropriate, there should be some small-scale collaborative work on a few specific themes of mutual interest and concern. These might be drawn from such topics as:

- Techniques of diagnostic testing
- Initial and in-service training of mathematics teachers
- The use of calculators in school mathematics
- Teaching mathematics to 5-7 year olds
- Applied mathematics in schools
- The role of mathematics in support of other subjects.

Collaboration might take the form of study-visits by individuals in each direction, the further exchange of information and research material, and perhaps some jointly-planned studies in both countries. The results of such cooperative work could then be fed into a further seminar, perhaps in 2 or 3 years time. These ideas will be considered in further detail, and proposals formulated, after the seminar.

SCHOOL VISITS

One morning of the Seminar was devoted to visits to three Moscow schools.

School 91

The primary classes of this school are used for experimental work by the Institute for Scientific Research into Content and Methods of Instruction. The British participant was able to visit two such lessons, in Classes 1 and 2. He was impressed by the attentiveness and keenness of the children, and the excellent relationship between teachers and pupils.

School 28

The majority of the British participants visited this Class 1-10 school. It was explained that it was only atypical in that it had an enhanced English programme; this meant that pupils started to learn English in Class 2 (age 8-9) rather than in Class 4 as in most schools.

We split up into three groups, to see a mathematics lesson in each of Classes 1, 3 and 10. We were also able to see a short sequence of English teaching at Class 10 level.

Special Boarding School for Mathematics and Science

The existence of specialised schools for the teaching of certain subjects is well known outside the Soviet Union. It is less well known that such schools only offer an enhanced programme in their special subjects from Class 8 (ages 14-15). One of the British participants was able to visit the Special Mathematics School associated with Moscow State University, and to observe two mathematics lessons at Class 9 level. One was on elementary probability; this topic does not feature in the curriculum applicable to all schools. The other lesson was on drawing graphs, including asymptotes, limits and the like. In this school, high standards of mathematics are achieved, and third-year undergraduates act as teachers' assistants, helping pupils with their difficulties, and so on.

In each of the schools visited, the British visitors were very impressed by the welcome they received from both teachers and pupils, who were invariably friendly and eager to explain the merits of their school.

PROGRAMME

Tuesday, September 21

- 1000 **Opening Session**
- Address by Academician I D Zverev, Vice-President of the USSR Academy of Pedagogical Sciences
- Problems of methods in introducing universal secondary education
V M MONAKHOV
- 1500 **Session 2. Major current problems in mathematics education**
- Problems of mathematics education in England
T J FLETCHER
- The basic trends in the improvement of mathematics teaching in the present day Soviet school
V V FIRSOV
- 1900 Visit to the Moscow State Ballet, in the Palace of Congresses

Wednesday, September 22

- 1000 **Session 3. Aims in mathematics education**
- "Mathematics in Employment 16-18": implications for the classroom
A FITZGERALD
- The aims of mathematics education A G HOWSON
- 1500 **Session 4. Achieving a balance in mathematics education**
- On the relationships between content and form in school mathematics
G V DOROFYEV
- The balance between pure and applied mathematics
F R WATSON
- 1800 Cocktail party given by Assistant Cultural Attache, British Embassy

Thursday, September 23

- 1000 **Session 5. Developmental work in mathematics education**
- Mathematics teaching projects in Great Britain
A J BISHOP
- Teaching and methodological materials for mathematics
N G MINDYK
- The School Mathematics Project (SMP) - a review
J HERSEY
- 1500 Visit to Exhibition of Economic Achievements

Friday, September 24

- 1000 **Session 6. How to achieve what is planned in mathematics education**
 Requirements for the mathematical training of school pupils
 A A GONCHAR
- Planning the mathematics curriculum for desired outcomes
 D A QUADLING
- 1500 **Session 7. The new geometry course in USSR**
 Content and methods of teaching geometry
 A V POGORELOV
- 1900 Meeting with the Editorial Board of 'Kvant'

Saturday, September 25

Sightseeing in Moscow

Sunday, September 26

- 1000 Visit to Archangelskoe country house museum, followed by picnic
- 1700 Visit to the Kremlin

Monday, September 27

- 1000 **Session 8. Some specific issues in mathematics education**
 Mathematics in the infant school (ages 5-7)
 E A ALBANY
- Planning the outcomes of mathematics education
 L V KUZNETSOVA
- Content and methods in the teaching of algebra and basic calculus
 J N MAKARYCHEV
- 1500 **Session 9. Methods of assessment in mathematics education**
 Examinations and assessment A W BELL
- Examinations and the curriculum J HERSEE
- 1900 Visit to the circus

Tuesday, September 28

- Morning Visits to schools
- Afternoon Visit to the Tretykovskaya Gallery
- Evening Dinner given by the British participants for the Soviet delegation.

Wednesday, September 29

1000 **Session 10. Training mathematics teachers in the USSR**

Requirements for the training of mathematics teachers
N E SHKIL

The pedagogic and methodological training of teachers of
mathematics R S CHERKASOV

The mathematical training of mathematics teachers
N J VILENKIN

1500 Closing session.

Evening Dinner given by the Soviet delegation.

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